SOME STRUCTURE THEOREMS FOR LATTICE-ORDERED GROUPS

BY

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To Reinhold Baer on his 60th birthday

1. **Introduction.** Throughout this paper $L=L(<, \cup, \cap, +)$ will always denote an l-group, and order will always mean linear order. An element $s\in L$ is *basic* if $0<s$ and $\{x\in L: 0<x\leqq s\}$ is ordered. For each $a\in L$, let L^a be the intersection of all convex subgroups of L that contain a. An element s of L is basic if and only if $0<s$ and L^s is ordered (Lemma 2.1). Each ordered convex subgroup of L is contained in a maximal convex chain of L. If R is a maximal convex chain of L that contains 0, then R is a convex ordered subgroup of L (Theorem 3.1). This generalizes the result of Jakubik [5] that a maximal chain of L that contains 0 and is convex, is a convex ordered subgroup of L. It follows that each basic element of L is contained in a unique maximal convex ordered subgroup of L, and any two such subgroups are disjoint.

A subset S of L is a *basis* for L if S is a maximum set of disjoint elements and each s in S is basic. L has a basis if and only if each strictly positive element of L is greater than or equal to at least one basic element (Theorem 5.1). In Theorem 5.2 we show that L has a basis provided that it satisfies the following condition.

(F) Each strictly positive element of L is greater than or equal to at most a finite number of disjoint elements.

One of the main results of this paper is that an l-group that satisfies (F) can be constructed from ordered groups by a sequence of cardinal and lexicographic extensions (Theorem 6.1). Each l-group with a basis contains a greatest l-ideal C that satisfies (F) (Theorem 6.2). The structure of C is then given by Theorem 6.1. We show in Theorem 6.3 that if L satisfies (F), then L is a topological group in its interval topology if and only if L is ordered.

If $S=\{a_\gamma: \gamma\in\Gamma\}$ is a basis for L, then each a_γ is contained in a maximal convex ordered subgroup A_γ of L, and the small cardinal sum $\sum + A_\gamma$ of these o-groups is an l-ideal of L. If T is any other basis for L, then T consists of elements from the A_γ; one and only one element from each A_γ. In Theorem 7.2 we prove that the following are equivalent.

(i) L has a basis and no maximal convex ordered subgroup of L is bounded from above.

(ii) There exists an l-isomorphism of L onto a sublattice of the large

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cardinal sum of the A_γ, and this sublattice contains the small cardinal sum of the A_γ.

In particular, L is Archimedean and has a basis if and only if L can be embedded in a large cardinal sum of subgroups of the real numbers (Theorem 7.3). L is a small cardinal sum of Archimedean ordered groups if and only if L is Archimedean and satisfies (F). We also show that if L is complete and has a basis, then L is l-isomorphic to the large cardinal sum of the A_γ if and only if every basis of L has an upper bound.

In §8 we explore the possibility of generalizing Theorem 6.1 in order to obtain a structure theorem for an arbitrary l-group with a basis. In §10 we obtain some results for l-groups that do not have a basis. In §11 the previous results are applied to commutative l-groups. Each abelian l-group L has a unique divisible closure D, and if L satisfies (F), then so does D. Applying Theorems 6.1, 7.2, and 7.3 to D we get a complete structure theorem because a lexico-extension of a divisible abelian l-group by a divisible abelian l-group is necessarily direct (Proposition 11.2).

The theory in §9 is entirely due to A. H. Clifford. Also Clifford was exposed to an earlier version of the proof of Theorem 6.1 and suggested improvements, many of which are incorporated in this proof. In particular, the original proof of Theorem 6.1 made use of the special case when L has a finite basis [4, Theorem 1] and this version does not.

2. **Cardinal sums, lexico-extensions and lexico-sums.** A subset S of an l-group L is *convex* if

(i) $a<x<b$ and $a, b\in S$ imply that $x\in S$, and

(ii) $a\cup 0\in S$ for all $a\in S$.

Clearly the intersection of convex subsets of L is a convex subset of L, and a set of positive elements of L is convex if and only if it satisfies (i). If S is a subgroup of L, then (i) is equivalent to

(i′) $0<x<b$ and $b\in S$ imply that $x\in S$.

A subset S of L is an *l-ideal* [2, p. 222] if and only if S is a normal convex subgroup of L. L is a *cardinal sum* of l-ideals A_γ for $\gamma\in\Gamma$ (notation $\sum\boldsymbol{+}A_\gamma$) if L is the small direct sum of the A_γ (notation $\sum\oplus A_\gamma$), and $a_{\gamma_1}+\cdots+a_{\gamma_n}\geqq 0$, where $a_{\gamma_i}\in A_{\gamma_i}$ and $\gamma_i\neq\gamma_j$ if $i\neq j$, if and only if $a_{\gamma_i}\geqq 0$ for $i=1,\cdots,n$. The corollaries to the next theorem show that some of the properties in this definition are redundant. If $L=\sum\boldsymbol{+}A_\gamma$, where the A_γ are ordered groups (notation o-groups), then it is easy to show that the A_γ are the maximal ordered l-ideals of L. In fact, in §7 we show that the A_γ are the maximal chains in L that contain 0 and are convex.

For any subset S of L, let $[S]$ denote the subgroup of L that is generated by S.

THEOREM 2.1. *Let L_γ for $\gamma\in\Gamma$ be convex subsemigroups of positive elements of L such that $L_\alpha\cap L_\beta=\{0\}$ if $\alpha\neq\beta$ and let A be the subsemigroup of L that is generated by the L_γ.*

(a) *$L_\alpha \cap$(subsemigroup generated by all the L_γ for $\gamma \neq \alpha$) $= \{0\}$ for all $\alpha \in \Gamma$, and $a_\alpha + a_\beta = a_\beta + a_\alpha$ for all $a_\alpha \in L_\alpha$ and $a_\beta \in L_\beta$ provided $\alpha \neq \beta$. Moreover, if $x = x_{\alpha_1} + \cdots + x_{\alpha_n}$, where $0 \neq x_{\alpha_i} \in L_{\alpha_i}$ and $\alpha_i \neq \alpha_j$ if $i \neq j$, then $x = x_{\alpha_1} \cup \cdots \cup x_{\alpha_n}$ and this representation is unique.*

(b) $[A] = \{a - b : a, b \in A\}$. *$[A]$ is convex and A is the convex subsemigroup of all positive elements of $[A]$.*

(c) $[A] = \sum + [L_\gamma]$.

COROLLARY I. *If $A_\gamma (\gamma \in \Gamma)$ is a convex subgroup of L and if the subgroup G of L generated by the A_γ is the small direct sum of the A_γ, then $G = \sum + A_\gamma$.*

This theorem and corollary are proven in [4] for finite Γ, and the extension to infinite Γ is immediate. In particular, if C is a convex semigroup of positive elements of L that contains 0, then $[C] = \{x - y : x, y \in C\}$, $[C]$ is convex and C is the semigroup of all positive elements of $[C]$. Conversely if S is a convex subgroup of L, then $S = \{a - b : a, b \in S^+\}$, where $S^+ = \{s \in S : s \geqq 0\}$. In particular, $L = \{a - b : a, b \in L^+\}$.

COROLLARY II. *If L is a small direct sum of subgroups A_γ for $\gamma \in \Gamma$, then the following are equivalent.*

(1) $L = \sum + A_\gamma$.

(2) *Each A_γ is convex.*

(3) *$a_{\gamma_1} + \cdots + a_{\gamma_n} \geqq 0$, where $a_{\gamma_i} \in A_{\gamma_i}$ and $\gamma_i \neq \gamma_j$ for $i \neq j$ if and only if $a_{\gamma_i} \geqq 0$ for $i = 1, \cdots, n$.*

For it follows from the definition of cardinal sum that (1) implies (2) and (3), and by Corollary I, (2) implies (1). The proof that (3) implies (2) is straightforward, and we omit it.

L is a *lexico-extension* of an l-group S (notation $L = \langle S \rangle$) if S is an l-ideal of L, L/S is an o-group, and each positive element in $L \backslash S$ exceeds every element in S. Trivially, $L = \langle L \rangle$, and $L = \langle 0 \rangle$ if and only if L is an o-group. Let S be an l-ideal of L. In Lemma 9.1 we show that $L = \langle S \rangle$ if and only if each nonzero element in L/S consists entirely of positive elements or entirely of negative elements. If $S \neq 0$, then $L = \langle S \rangle$ if and only if each positive element in $L \backslash S$ exceeds every element in S [4]. Let S be an l-group, T an o-group, and $L = S \oplus T$. Define that $s + t \in L$ is positive if $t > 0$ or $t = 0$ and $s \geqq 0$. Then $L = \langle S \rangle$ and we say that L is a *direct lexico-extension* of S. In §11 we prove that if L is divisible and abelian, and if $L = \langle S \rangle$, then L is a direct lexico-extension of S. In [4] examples are given of lexico-extensions that are not direct. Also, see the example in this paper after Theorem 2.3.

Let $A_1, \cdots, A_n$ be o-groups; then by a finite alternating sequence of cardinal summations and lexico-extensions we can construct l-groups from the A_i in which each A_i is used exactly once to make a cardinal extension and the o-groups used to make the lexico-extensions are arbitrary. We call such groups *lexico-sums* of the A_i. For example, if $n = 3$, then there are two ways of constructing lexico-sums of A_1, A_2, A_3 in this order, namely, $\langle A_1 + \langle A_2 + A_3 \rangle \rangle$

and $\langle\langle A_1 \boxplus A_2\rangle \boxplus A_3\rangle$. A subset $\{a_\gamma : \gamma \in \Gamma\}$ of L is *disjoint* or *the elements a_γ are disjoint* if each $a_\gamma > 0$ and $a_\alpha \cap a_\beta = 0$ for all $\alpha \neq \beta$. In particular, the null set $\square$ is disjoint. In [4] the following theorem is proven.

THEOREM 2.2. *Suppose that L contains n disjoint elements $a_1, \cdots, a_n$ but does not contain $n+1$ such elements. Let $L_i = \{x \in L : x \cap a_j = 0$ for all $j \neq i\}$. Then the $[L_i]$ are o-groups and L is a lexico-sum of the $[L_i]$.*

In Theorem 6.1 we generalize this result to the case where each positive element of L exceeds at most a finite number of disjoint elements, and we prove this result without using Theorem 2.2.

COROLLARY I. *L is a lexico-sum of n ordered subgroups if and only if L contains n disjoint elements but does not contain $n+1$ such elements.*

COROLLARY II. *Suppose that L contains n disjoint elements but not $n+1$ such elements. Then the following are equivalent.*

(a) *L is a proper lexico-extension of an l-ideal.*

(b) *For each proper convex subgroup C of L there exists an element a in L^+ such that $a > C$.*

Proof. Suppose that L satisfies (b). Then since L is a lexico-sum of a finite number of o-groups, either $L = \langle I\rangle$ for some l-ideal $I \neq L$ or $L = A \boxplus B$, where A and B are nonzero l-ideals of L. In the latter case no a in L^+ exceeds A. Conversely suppose that $I \neq L$ is an l-ideal of L and $L = \langle I\rangle$. If C is a proper convex subgroup of L, then either $C \subseteq I$ or $C \supset I$. If $C \subseteq I$, then each $a \in L^+ \backslash I$ exceeds C. If $C \supset I$, then C/I is a proper convex subgroup of the o-group L/I. Pick an X in L/I that exceeds every element in C/I, and let $c \in C$. Then $X = I + x > I + c$ and hence $I + x - c > I$. It follows that $x - c > 0$, and hence $x > C$.

Note that (a) implies (b) in Corollary II with no restrictions on L, but that the converse is, in general, not true.

For each $a \in L$, let L^a be the intersection of all convex subgroups of L that contain a. Thus L^a is the smallest convex subgroup of L that contains a.

LEMMA 2.1. *Let $0 < y$ be an element in L.*

(i) *L^y does not contain any element disjoint from y.*

(ii) *L^y is ordered if and only if $\{x \in L : 0 < x \leqq y\}$ is ordered.*

Proof. (i) If $0 < a \in L^y$ and $y \cap a = 0$, then $L_a = \{x \in L : x \cap a = 0\}$ is a convex subsemigroup of positive elements that contain 0 and y but not a. For if $0 < z \leqq x \in L_a$, then $0 \leqq z \cap a \leqq x \cap a = 0$ and so $z \in L_a$. Thus L_a is convex. See [2, p. 219] for a proof that L_a is a semigroup. By Theorem 2.1, $[L_a]$ is convex and L_a is the semigroup of all positive elements of $[L_a]$. Thus $[L_a] \supseteq L^y$ and $a \notin [L_a]$, a contradiction.

(ii) Let $A = \{x \in L : 0 < x \leqq y\}$. Since L^y is convex it contains A. Thus if L^y is ordered, then so is A. Conversely suppose that A is ordered, and let

$P=\{x\in L^y : x>0\}$. It suffices to show that P does not contain a pair of disjoint elements. Consider a, $b\in P$. By (i) $a\cap y>0$ and $b\cap y>0$. Thus $a\cap y$ and $b\cap y$ belong to A which is ordered, and so $a\cap b\cap y>0$. Therefore $a\cap b>0$, and hence L^y is ordered. We say that L *has finite rank* if it contains only a finite number of convex subgroups, and that a lexico-extension L of an l-group S *is of finite rank* if L/S has finite rank.

THEOREM 2.3. *L has finite rank if and only if L is a lexico-sum of a finite number of o-groups each of which has finite rank and the lexico-extensions used in the construction of L are also of finite rank.*

Proof. Suppose that L has finite rank m. Assume (by way of contradiction) that $x_1, x_2, \cdots, x_{m+1}$ are disjoint elements in L. Then by Lemma 2.1, $x_i \notin L^{x_j}$ for $i\neq j$. Thus $L^{x_1}, \cdots, L^{x_{m+1}}$ are distinct convex subgroups of L, a contradiction. It follows that there exists a positive integer $n\leqq m$ such that L contains n disjoint elements but not $n+1$ such elements. By Theorem 2.2, L is a lexico-sum of n o-groups $A_1, \cdots, A_n$. Each convex subgroup of each A_i is also a convex subgroup of L. Therefore each A_i has finite rank. If B and C are convex subgroups of L and $B=\langle C\rangle$, then there exists a 1-1 correspondence between the convex subgroups of B/C and the convex subgroups of L that are contained in B and contain C. Therefore B/C has finite rank.

Conversely suppose that L is a lexico-sum of o-groups $A_1, \cdots, A_n$ each of which has finite rank and the lexico-extensions used in the construction of L are also of finite rank. Then $L=\langle X\boldsymbol{+}Y\rangle$, where X is a lexico-sum of $A_1, \cdots, A_s$ and Y is a lexico-sum of $A_{s+1}, \cdots, A_n$ for a suitable ordering of the subscripts, and $L/(X\boldsymbol{+}Y)$ has finite rank. Now any convex subgroup of L is contained in $X\boldsymbol{+}Y$ or contains $X\boldsymbol{+}Y$. By induction X and Y have finite rank. Therefore $X\boldsymbol{+}Y$ and hence L has finite rank.

REMARKS. This includes as a special case Birkhoff's result: If the lattice of l-ideals of a commutative l-group G is finite, then either $G=B\boldsymbol{+}C$ or G contains a maximal l-ideal that contains every other proper l-ideal [2, p. 237]. If L contains only a finite number of l-ideals and L is nonabelian, then L may contain an infinite number of disjoint elements. For example, let I be the o-group of integers, and for each $i\in I$ let $I_i=I$, and let $N=\sum_{i\in I}\boldsymbol{+}I_i$. For $(\cdots, a_i, \cdots)$ in N and j in I define that $(\cdots, a_i, \cdots)r(j) =(\cdots, a_{i+j}, \cdots)$. That is, the $(i+j)$th component is replaced by the ith component. Let $G=I\times N$ and define $(i, a)+(j, b)=(i+j, ar(j)+b)$ and (i, a) positive if $i>0$ or $i=0$ and a is positive in N. It follows that $G=\langle N\rangle$ and that N is the only proper l-ideal of G. But clearly G contains an infinite number of disjoint elements. Note that G is finitely generated.

3. Ordered convex subgroups and maximal convex chains.

LEMMA 3.1. *If A and B are convex ordered subgroups of L, then $A\subseteq B$ or $A\supseteq B$ or $A\cap B=0$.*

Proof. Suppose that $A \not\supseteq B$, $A \not\subseteq B$ and $A \cap B \neq 0$. Then there exist $0 < a \in A \setminus B$ and $0 < b \in B \setminus A$. $a \cap b \in A \cap B$ because A and B are convex. If $0 < c \in A \cap B$, then $c < a$ and $c < b$ because $A \cap B$ is an ordered convex subgroup of the o-groups A and B. Thus $c \leqq a \cap b$. But this means that $a \cap b$ is a maximal element in the nonzero o-group $A \cap B$ which is impossible.

COROLLARY. *If $0 \neq A$ is a convex ordered subgroup of L, then there exists a greatest convex ordered subgroup of L that contains A.*

For by our lemma the set of all ordered convex subgroups that contain A is ordered by inclusion. Hence the join of this set is the desired group. Note that every chain S of elements of L that contains 0 satisfies

(ii) $0 \cup a \in S$ for all $a \in S$.

Thus such a chain is convex if and only if it satisfies

(i) $a < x < b$ and $a, b \in S$ imply that $x \in S$.

THEOREM 3.1. *If R is a maximal convex chain of L that contains* 0, *then R is a convex ordered subgroup of L.*

Proof. Let $R^+ = \{x \in R: x \geqq 0\}$ and $R^- = \{x \in R: x \leqq 0\}$. Then $R = R^+ \cup R^-$ because R is a chain that contains 0. If $0 < y \in R$, then $\{x \in L: 0 < x \leqq y\}$ is ordered, hence by Lemma 2.1, L^y is ordered. Thus $S = \bigcup_{y \in R^+} L^y$ is a convex o-subgroup of L and $R^+ \subseteq S$, because S is the join of a chain of convex o-subgroups of L. Similarly, $T = \bigcup_{y \in -R^-} L^y$ is a convex o-subgroup of L and $R^- \subseteq T$. If $R^+ = 0$, then $R = T$ and hence $R = 0$, and if $R^- = 0$, then $R = S$ and hence $R = 0$. Suppose that $R \neq 0$. By Lemma 3.1, $S \subseteq T$ or $T \subseteq S$ or $S \cap T = 0$. If $S \subseteq T$, then $R = R^+ \cup R^- \subseteq T$, and since R is maximal, $R = T$. Similarly, if $T \subseteq S$, then $R = S$.

Finally assume that $S \cap T = 0$. Then $R^+ \cap -R^- = 0$. Pick $0 < p \in R^+$ and $0 > n \in R^-$, and let $z = n + p$. Then $n = z - p < z < -n + z = p$. Thus $z \in R$. If $z \in R^+$, then $0 < -n = p - z \leqq p$, and hence $-n$ belongs to $R^+ \cap -R^- = 0$, a contradiction. If $z \in R^-$, then $n \leqq -z + n = -p < 0$, hence $-p \in R^-$ and p belongs to $R^+ \cap -R^-$, a contradiction. Therefore $S \cap T \neq 0$. It follows that $R = S = T$.

COROLLARY I (JAKUBIK). *If R is a maximal chain of L that is convex and contains* 0, *then R is a convex ordered subgroup of L.*

Jakubik [5] also proves that if R is a maximal chain of L that is convex and contains 0, then R is a direct summand of L. See Lemma 7.1 in this paper for a short proof of this result.

COROLLARY II. *Each convex chain C of L that properly contains* 0 *is contained in a unique maximal convex ordered subgroup of L, namely, the maximal convex chain M of L that contains C.*

The uniqueness of M follows from the corollary to Lemma 3.1.

4. Bases for L-groups. An element s of L is *basic* if $0 < s$ and $\{x \in L: 0 < x \leqq s\}$ is ordered. Thus by Lemma 2.1, s is basic if and only if $0 < s$ and L^s is an o-group. A subset S of L is a *basis* if

(i) S is a maximal set of disjoint elements of L, and

(ii) each s in S is basic.

The null set is a basis for the one element l-group. If $0 < s \in L$ and L is an o-group, then $\{s\}$ is a basis of L.

LEMMA 4.1. *A nonvoid subset S of L is a basis if and only if S is disjoint and $(S \backslash \{s\}) \cup \{x, y\}$ is not disjoint for any $s \in S$ and $x \neq y$ in $(L \backslash S) \cup \{s\}$.*

Proof. If S is a basis of L and $(S \backslash \{s\}) \cup \{x, y\}$ is disjoint for some s in S and $x \neq y$ in $(L \backslash S) \cup \{s\}$, then $x \cap s > 0$ and $y \cap s > 0$ because S is a maximal set of disjoint elements. Thus $x \cap s$ and $y \cap s$ belong to $\{z \in L: 0 < z \leqq s\}$ which is ordered. Therefore

$$0 < x \cap s \cap y \cap s = x \cap y \cap s = 0 \cap s = 0,$$

a contradiction. Conversely suppose that S is disjoint and satisfies the condition in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in L \backslash S$, then for any $s \in S$, $(S \backslash \{s\}) \cup \{s, x\}$ is disjoint and $s \neq x$, a contradiction. If $s \in S$ and s is not basic, then there exist strictly positive elements x and y in $L \backslash S$ such that $x \cap y = 0$, $s > x$ and $s > y$. Thus $(S \backslash \{s\}) \cup \{x, y\}$ is disjoint, a contradiction. Therefore S is a basis of L.

Theorem 2.2 is a structure theorem for l-groups with finite bases. *For the remainder of this section we assume that $S = \{a_\gamma: \gamma \in \Gamma\}$ is a basis of L.*

LEMMA 4.2. *If π is an o-permutation of L such that $0\pi = 0$, then $S\pi$ is a basis of L.*

Proof. $a_\alpha \pi \cap a_\beta \pi = (a_\alpha \cap a_\beta)\pi = 0\pi = 0$ if $\alpha \neq \beta$ and $a_\alpha \pi > 0$ for all $\alpha \in \Gamma$. Thus $S\pi$ is a disjoint set and in fact a maximal disjoint set. Clearly π maps basic elements onto basic elements.

For each $\gamma \in \Gamma$ let $L_\gamma = \{x \in L: x \cap a_\beta = 0$ for all $\beta \neq \gamma\}$ and let $B_\gamma = \{x \in L: x \cap a_\gamma = 0\}$. The following seven propositions are proven in [4] for finite Γ and the proofs given there can easily be extended to the case where Γ is infinite by using Theorem 2.1 and Lemmas 4.1 and 4.2.

4.1. L_γ is an ordered convex subsemigroup of positive elements of L. $[L_\gamma] = \{x - y: x, y \in L_\gamma\}$ is a convex ordered subgroup of L and $L_\gamma = \{x \in [L_\gamma]: x \geqq 0\} = [L_\gamma]^+$.

4.2. Let A be the subsemigroup of L that is generated by the L_γ. $[A] = \sum\!+ [L_\gamma] = \{a - b: a, b \in A\}$, $[A]$ is convex and $A = \{x \in [A]: x \geqq 0\}$.

4.3. Pick $0 < b_\gamma \in L_\gamma$ and define $H_\gamma = \{x \in L: x \cap b_\beta = 0$ for all $\beta \neq \gamma\}$. Then $L_\gamma = H_\gamma$ for all γ and $\{b_\gamma: \gamma \in \Gamma\}$ is a basis for L. In particular, if for each $\gamma \in \Gamma$ we pick a $b_\gamma \in L$ such that $0 < b_\gamma \leqq a_\gamma$, then $\{b_\gamma: \gamma \in \Gamma\}$ is a basis of L.

4.4. B_γ is a convex subsemigroup of positive elements of L. Let B^γ be the subsemigroup generated by B_γ and L_γ. The $[B^\gamma]=[L_\gamma]\boldsymbol{+}[B_\gamma]$, $[B^\gamma]$ is convex and $B^\gamma=\{x\in[B^\gamma]: x\geqq 0\}$. B_γ does not depend upon the particular choice of the a_γ in the L_γ. $\{a_\alpha: \alpha\in\Gamma \text{ and } \alpha\neq\gamma\}$ is a basis for $[B^\gamma]$.

Let Δ be a nonvoid subset of Γ, and let

$$L_\Delta = \{x \in L: x \cap a_\gamma = 0 \text{ for all } \gamma \in \Gamma\backslash\Delta\}.$$

4.5. L_Δ is a convex subsemigroup of positive elements of L that contains $\sum_{\delta\in\Delta}\boldsymbol{+}L_\delta$ and L_Δ is independent of the particular choice of the a_γ in the L_γ. $[L_\Delta\cup(\bigcup_{\gamma\in\Gamma-\Delta} L_\gamma)]=[L_\Delta]\boldsymbol{+}\sum_{\gamma\in\Gamma-\Delta}\boldsymbol{+}[L_\gamma]$. If Δ is a finite set and $0<a \in L_\Delta\backslash\sum_{\delta\in\Delta}\boldsymbol{+}L_\delta$, then there exist $\alpha\neq\beta$ in Δ such that $a>L_\alpha+L_\beta$.

The first example in §8 shows that the last proposition does not hold for infinite subsets Δ of Γ.

4.6. If T is a basis of L, then $T\subseteq A$. In fact, each $t\in T$ belongs to one L_γ and each L_γ contains exactly one $t\in T$. In particular, S and T have the same cardinality. (This proposition follows from Theorems 5.3 and 5.4 in this paper.)

4.7. $[A]$ is invariant with respect to l-automorphisms of L. Thus $[A]$ is an l-ideal of L and $L/[A]$ is an l-group.

Suppose that L is a subgroup of an l-group G and that L is a lattice with respect to the partial order induced by G. Also suppose that for each $0<g\in G$ there exists an $x\in L$ such that $0<x\leqq g$. Then the basis S of L is also a basis for G. Conversely if $\{g_\gamma: \gamma\in\Gamma\}$ is a basis of G and if for each $\gamma\in\Gamma$ we pick a $b_\gamma\in L$ such that $0<b_\gamma\leqq g_\gamma$, then by Proposition 4.3, $Q=\{b_\gamma: \gamma\in\Gamma\}$ is a basis for G. It follows that Q is a maximum disjoint subset of L and that each b_γ is basic in L. Therefore Q is a basis of L. In particular, let D be the completion of L by nonvoid cuts (see [2, p. 229] or [1]), and let G be the group of units of the semigroup D. Then G is an l-group that contains L and satisfies the above hypotheses.

5. **Independent subsets of L-groups.** A subset S of L is *independent* if S is a disjoint set and each s in S is basic.

Lemma 5.1. *A subset $S=\{a_\gamma: \gamma\in\Gamma\}$ of L is independent if and only if $0<a_\gamma$ and L^{a_γ} is an o-group for each $\gamma\in\Gamma$, and $[\bigcup_{\gamma\in\Gamma} L^{a_\gamma}]=\sum\boldsymbol{+}L^{a_\gamma}$.*

Proof. If S satisfies the conditions in the lemma, then by Lemma 2.1 each a_γ is basic. $a_\alpha\cap a_\beta\in L^{a_\alpha}\cap L^{a_\beta}$ because L^{a_α} and L^{a_β} are convex. Thus if $\alpha\neq\beta$, then $a_\alpha\cap a_\beta=0$, and so S is independent. Conversely if S is independent, then $0<a_\gamma$ and L^{a_γ} is an o-group for each γ in Γ because each a_γ is basic. $L^{a_\alpha}\cap L^{a_\beta}=0$ if $\alpha\neq\beta$, for otherwise by Lemma 3.1, $L^{a_\alpha}\subseteq L^{a_\beta}$ or $L^{a_\beta}\subseteq L^{a_\alpha}$, and hence $a_\alpha\cap a_\beta\neq 0$. Thus by Theorem 2.1, $[\bigcup_{\gamma\in\Gamma} L^{a_\gamma}]=\sum\boldsymbol{+}L^{a_\gamma}$.

Lemma 5.2. *Each independent subset of L is contained in a maximal independent subset of L. In particular, there exists a maximal independent subset T of L, and $T=\square$ if and only if L contains no basic elements, that is, if and only if* *every strictly positive element of L is greater than a pair of disjoint elements.*

This follows immediately from the fact that the null set $\square$ is an independent set and an easy application of Zorn's lemma.

THEOREM 5.1. *L has a basis if and only if L satisfies*

$$(*)\ \textit{each } 0<x\in L \textit{ exceeds at least one basic element.}$$

Every basis of L is a maximal independent set and every maximal independent subset of L is a basis provided that L has a basis.

Proof. If $L=0$, then the null set is a basis for L and the theorem is trivial. Assume that $L\neq 0$. Let $S=\{0<a_\gamma:\gamma\in\Gamma\}$ be a basis for L, and consider $0<x\in L$. There exists a $\gamma\in\Gamma$ such that $x\cap a_\gamma>0$, for otherwise S is not a maximal set of disjoint elements. But this means that $0<x\cap a_\gamma\leqq x$ and $x\cap a_\gamma\in[L_\gamma]$ which by Proposition 4.1 is a convex ordered subgroup of L. Therefore $L^{x\cap a_\gamma}$ is ordered and hence $x\cap a_\gamma$ is basic. Thus L satisfies (*), and clearly S is a maximal independent subset of L.

Conversely suppose that L satisfies (*). By Lemma 5.2 there exists a maximal independent subset $T=\{0<a_\gamma:\gamma\in\Gamma\}$ of L and $T\neq\square$. We wish to show that T is a basis for L. It suffices to show that T is a maximal set of disjoint elements. Suppose (by way of contradiction) that there exists $0<x\in L$ such that $x\cap a_\gamma=0$ for all $\gamma\in\Gamma$. Then by (*) there exists $y\in L$ such that $0<y\leqq x$ and y is basic. $0\leqq y\cap a_\gamma\leqq x\cap a_\gamma=0$. Therefore $T\cup\{y\}\supset T$ and $T\cup\{y\}$ is an independent subset of L, but this is contrary to our choice of T.

COROLLARY I. *If L has a basis and C is a convex subgroup of L, then C has a basis.*

For if L satisfies (*), then so does C. The first example in §8 shows that if L has a basis and if C is an l-ideal of L, then L/C need not have a basis.

COROLLARY II. *If L has a basis and T is an independent subset of L, then T is contained in a basis of L.*

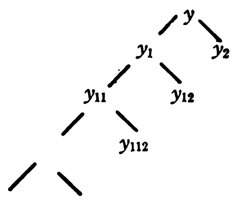
For by Lemma 5.2, T is contained in a maximal independent subset S of L, and by Theorem 5.1, S is a basis for L.

We shall frequently restrict our attention to l-groups that satisfy:

(F) *Each $0<a\in L$ is greater than at most a finite number of disjoint elements.*

THEOREM 5.2. *If L satisfies* (F), *then L has a basis.*

Proof. Assume (by way of contradiction) that $0<y\in L$ and that no basic element is contained in $\{x\in L:0<x\leqq y\}$. Then $\{z\in L:0<z\leqq x\}$ is not ordered for any $x\leqq y$. But this means that each $x\leqq y$ is greater than a pair of disjoint elements. Thus we have the following diagram of strictly positive elements in L.



$$y_{11} \cap y_{12} = 0$$

$$y_{111} \cap y_{112} = 0$$

It follows that $y_{12}, y_{112}, y_{1112}, \cdots$ is an infinite disjoint set of elements each of which is less than y, a contradiction. Therefore y exceeds at least one basic element, and hence by Theorem 5.1, L has a basis.

COROLLARY I. *L is a lexico-sum of a finite number of ordered subgroups if and only if L does not contain an infinite disjoint set.*

Proof. If L does not contain an infinite disjoint set, then clearly L satisfies (F), and hence has a basis. This basis being a disjoint set is necessarily finite. Therefore by Theorem 2.2, L is a lexico-sum of a finite number of ordered subgroups. The converse follows immediately from Corollary I of Theorem 2.2.

COROLLARY II. *If $0 < a \in L$ is greater than at most a finite number of disjoint elements, then L^a is a lexico-sum of a finite number of ordered subgroups.*

Proof. Suppose (by way of contradiction) that $x_1, x_2, \cdots$ is an infinite set of disjoint elements of L^a. Then by Lemma 2.1, $x_i \cap a \neq 0$ for all i, and if $i \neq j$, then $x_i \cap a \neq x_j \cap a$ because $(x_i \cap a) \cap (x_j \cap a) = x_i \cap x_j \cap a = 0 \cap a = 0$. Thus $\{x_i \cap a : i = 1, 2, \cdots\}$ is an infinite set of disjoint elements of L each of which is less than or equal to a, a contradiction. This corollary now follows from Corollary I.

COROLLARY III. *If L satisfies* (F), *then each L^a $(0 < a \in L)$ and each convex subgroup C of L that is bounded from above is a lexico-sum of a finite number of ordered subgroups.*

Proof. If C is a convex subgroup of L, $a \in L$ and $a > C$, then $L^a \supseteq C$. But by Corollary II, L^a and hence C has the desired structure.

Let $T = \{a_\gamma : \gamma \in \Gamma\}$ be a maximal independent subset of L. By Lemma 5.1, for each γ in Γ, $a_\gamma > 0$ and L^{a_γ} is an o-group. Thus by the corollary to Lemma 3.1, for each $a_\gamma \in T$ there exists a greatest ordered convex subgroup A_γ of L that contains a_γ. Let $M = \{A_\gamma : \gamma \in \Gamma\}$ if $T \neq \square$ and let $M = \{0\}$ if $T = \square$.

THEOREM 5.3. *Let G be the subgroup of L that is generated by the A_γ in M. Then $G = \sum + A_\gamma$ and G is invariant with respect to all l-automorphisms of L. M is the set of all maximal ordered convex subgroups of L. In fact, M is the set of all maximal convex chains in L that contain* 0.

Proof. If $T=\square$, then by Lemma 5.2, L contains no basic elements, and hence by Lemma 2.1, $\{0\}$ is the only convex ordered subgroup of L. Suppose that $T\neq\square$. If $\alpha\neq\beta$, then $A_\alpha\cap A_\beta=0$. Otherwise by Lemma 3.1 $A_\alpha\subseteq A_\beta$ or $A_\beta\subseteq A_\alpha$, but then $a_\alpha\cap a_\beta\neq 0$. Thus by Theorem 2.1, $G=\sum\boldsymbol{+}A_\gamma$. Let Q be a maximal ordered convex chain of L that contains 0. Since L contains basic elements, $Q\neq 0$. If $Q\cap A_\gamma=0$ for all $\gamma\in\Gamma$, then by Theorem 2.1, $Q\boldsymbol{+}\sum\boldsymbol{+}A_\gamma$. But then $T\cup\{q\}$ is an independent subset for each $0<q\in A$, a contradiction. Therefore there exists a $\gamma\in\Gamma$ such that $Q\cap A_\gamma\neq 0$ and hence by Lemma 3.1 (since Q and A_γ are maximal) $Q=A_\gamma$. Therefore M is the set of all maximal ordered convex chains of L that contains 0. Finally if π is an l-automorphism of L, then clearly $A_\gamma\pi\in M$ for all $\gamma\in\Gamma$. Thus $(\sum\boldsymbol{+}A_\gamma)\pi=\sum\boldsymbol{+}A_\gamma$.

Corollary I. *If $\square\neq U=\{c_\delta\colon \delta\in\Delta\}$ is a maximal independent subset of L, then there exists a* 1-1 *mapping σ of Δ onto Γ such that $c_\delta\in A_{\delta\sigma}$.*

For each c_δ determines a unique greatest ordered convex chain C_δ that contains it and $C_\delta=A_\gamma$ for some $\gamma\in\Gamma$. Thus a maximal independent set is a set of strictly positive elements in $\bigcup_{\gamma\in\Gamma}A_\gamma$ that contains one and only one element from each A_γ, or the null set if $T=\square$.

Corollary II. *If C is an ordered convex subgroup of L and $T\neq\square$, then $C\subseteq A_\gamma$ for some $\gamma\in\Gamma$.*

For by the corollary to Lemma 3.1, C is contained in a maximal convex subgroup of L.

Theorem 5.4. *If $\square\neq T=\{a_\gamma\colon\gamma\in\Gamma\}$ is a maximal independent subset of L and L has a basis, then T is a basis and hence determines $[A]=\sum\boldsymbol{+}[L_\gamma]$, where $L_\gamma=\{x\in L\colon x\cap a_\beta=0$ for all $\beta\neq\gamma\}$ and A is the subsemigroup of L generated by the L_γ. Moreover, $A_\gamma=[L_\gamma]$ for all $\gamma\in\Gamma$.*

Proof. By Theorem 5.1, T is a basis. If $x\in A_\gamma^+$, then $x\cap a_\alpha=0$ for all $\alpha\neq\gamma$ because $[\bigcup_{\gamma\in\Gamma}A_\gamma]=\sum\boldsymbol{+}A_\gamma$. Thus $x\in L_\gamma$ and hence $A_\gamma^+\subseteq L_\gamma$. But this means that $A_\gamma\subseteq[L_\gamma]$ and so $A_\gamma=[L_\gamma]$ because A_γ is a maximal convex ordered subgroup of L, and $[L_\gamma]$ is a convex ordered subgroup of L (Proposition 4.1).

We shall call $[A]$ the *basis group* of L. If $T=\square$, then let $[A]=\{0\}$. Proposition 4.6 follows at once from Theorems 5.3 and 5.4. In particular, $[A]$ and the A_γ are independent of the particular choice of a basis. In fact, if $L\neq 0$ and L has a basis, then this basis is a subset of $\bigcup_{\gamma\in\Gamma}A_\gamma$ that contains one and only one element from each A_γ. If $L=0$, then the null set is a basis.

Theorem 5.5. *If L has a finite basis $a_1,\cdots,a_n$ and B is an l-ideal of L, then L/B has a basis of n or fewer elements.*

Proof. If $n=1$, then L and L/B are o-groups and the theorem is obvious. Suppose that $n>1$ and that the theorem is true for all l-groups that contain fewer than n disjoint elements. By Theorem 2.2, $L=\langle U\boxplus V\rangle$ where $a_1, \cdots, a_s \in U$ and $a_{s+1}, \cdots, a_n \in V$ and $1 \leqq s < n$. Let $t=n-s$. Suppose (by way of contradiction) that $Y_i = B+y_i$ are disjoint elements in L/B for $i=1, \cdots, n+1$; then $0<y_i \in L\backslash B$ and $y_i \cap y_j \in B$ for $i \neq j$.

If there exists a $y_j \in L\backslash(U\boxplus V)$, then $y_j > U\boxplus V$ and hence all the $y_i > U\boxplus V$. Therefore $y_i \cap y_j > U\boxplus V$ and $y_i \cap y_j \in B$ for $i \neq j$. Thus since $U\boxplus V$ and B are convex, $B \supseteq U\boxplus V$ and hence L/B is an o-group, which is impossible. Therefore all of the y_i belong to $U\boxplus V$. For each i, $y_i = y_{i1}+y_{i2} = y_{i1} \cup y_{i2}$, where $y_{i1} \in U$ and $y_{i2} \in V$.

$$y_i \cap y_j = (y_{i1} \cup y_{i2}) \cap (y_{j1} \cup y_{j2}) = (y_{i1} \cap y_{j1}) \cup (y_{i2} \cap y_{j2}).$$

Since U, V, and B are convex $y_{i1} \cap y_{j1} \in U \cap B$ and $y_{i2} \cap y_{j2} \in V \cap B$, and hence $y_i \cap y_j \in (U\cap B)+(V\cap B)$. Since $y_i \notin B$, either $y_{i1} \notin B$ or $y_{i2} \notin B$. At most s of the y_{i1} do not belong to B, for otherwise $U/(U\cap B)$ contains more than s disjoint elements, and this contradicts our induction hypothesis. Similarly, at most t of the y_{i2} do not belong to B. Therefore at most $s+t$ of the y_i do not belong to B, and hence $n=s+t \geqq n+1$, a contradiction. Therefore L/B has a basis of n or fewer elements.

COROLLARY. *If L has a finite basis of n elements and B is an l-ideal of L that contains the basis group $[A]$, then L/B has a basis of fewer than n elements.*

This follows from the proof of Theorem 5.5 except that we let the induction hypothesis be: $U/(U\cap B)$, $(V/(V\cap B))$ contain less than $s(t)$ disjoint elements.

6. Small lexico-sums of o-groups.

LEMMA 6.1. *If $L=A\boxplus B=C\boxplus D$, where A, B, C, D are l-ideals of L and $A\supseteq C$, then $B\subseteq D$. In particular, if $L=A\boxplus B=A\boxplus D$, then $B=D$.*

Proof. If $0<b\in B$, then $b=c+d$, where $0\leqq c\in C\subseteq A$ and $0\leqq d\in D$. Thus $0\leqq c\leqq b$ and since B is convex, $c\in B\cap A=0$. Therefore $B^+\subseteq D^+$ and hence $B\subseteq D$.

LEMMA 6.2. *Let B be a convex subgroup of L and let $B^*=\{x\in L^+ : x\cap B^+=0\}$, where $x\cap B^+=\{x\cap b : b\in B^+\}$.*

(i) *B^* is a convex subsemigroup of L, $[B^+\cup B^*]=B\boxplus[B^*]$, and $[B^*]$ and $B\boxplus[B^*]$ are convex.*

(ii) *If B is an l-ideal of L, then so are $[B^*]$ and $B\boxplus[B^*]$.*

(iii) *If B is a proper lexico-extension of an l-ideal I of B ($I=0$ being allowed), then $B^+ + B^* = \{x\in L^+ : x$ does not exceed every element of $B\}$.*

Proof. (i) B^* is the intersection of the convex subsemigroups $L_b = \{x\in L : x\cap b=0\}$, where $b\in B^+$, and hence B^* is a convex subsemigroup of positive elements that contains 0. Clearly $B^*\cap B^+=0$. Thus by Theorem 2.1 $[B\cup B^*]=B\boxplus[B^*]$, and $[B^*]$ and $B\boxplus[B^*]$ are convex. (ii) Let $b\in B^+$,

$x\in B^*$ and $y\in L$. Then $0=-y+(x\cap b)+y=(-y+x+y)\cap(-y+b+y)$. Thus if B is normal, then $0=(-y+x+y)\cap c$ for all $c\in B^+$ and hence $[B^*]$ is an l-ideal. (iii) Clearly no element of B^++B^* exceeds every element of B. Assume conversely that $c\in L^+$ and that c does not exceed every element of B. Then there exists an element b in $B^+\backslash I^+$ such that $c\ngeqq b$. $2b=2b\cap c+u$, $c=2b\cap c+v$ and $u\cap v=0$. Since $0\leqq u\leqq 2b$ and B^+ is convex, $u\in B^+$. If $u\in I^+$, then $b>u$ and hence $2b\cap c+u=2b>b+u$, and so $c\geqq 2b\cap c\geqq b$, a contradiction. Therefore $u\in B^+\backslash I^+$. For each x in B^+, $(x\cap v)\cap u=x\cap(u\cap v)=x\cap 0=0$. Thus since B is a proper lexico-extension of I it follows that $x\cap v=0$. Therefore $c=2b\cap c+v\in B^++B^*$.

We say that an l-group L is a small *lexico-sum of o-groups* C^1_γ $(\gamma\in\Gamma)$ if there exists a finite or infinite sequence C^n $(n=1, 2, \cdots)$ of l-ideals of L such that

(i) $C^1\subseteq C^2\subseteq\cdots$, and $\bigcup_{i=1}^\infty C^i=L$;

(ii) $C^1=\sum_{\gamma\in\Gamma}\boldsymbol{+}C^1_\gamma$;

(iii) for $n>1$, $C^n=\sum_{\gamma\in\Gamma_n}\boldsymbol{+}C^n_\gamma$, where each C^n_γ is a convex subgroup of L and either a nontrivial lexico-extension of a finite cardinal sum of two or more of the components C^{n-1}_β or else C^n_γ is equal to one of the C^{n-1}_β.

Suppose that L is a small lexico-sum of o-groups C^1_γ $(\gamma\in\Gamma)$. Then clearly each positive element of L is greater than at most a finite number of disjoint elements. Thus by Theorem 5.2, L has a basis, and hence a basis group $[A]$. It follows immediately that the C^1_γ are the maximal convex chains of L, and hence $C^1=[A]$. Also it is fairly easy to show that for each positive integer n

(v) $(C^n_\alpha)^+=\{x\in L^+: x\cap(C^n_\beta)^+=0$ for all $\beta\neq\alpha$ in $\Gamma_n\}$;

(vi) no strictly positive element in L is disjoint to every C^n_γ;

(vii) $(C^n)^+=\{x\in L^+: x\not> C^n_\alpha$ for any $\alpha\in\Gamma_n\}$;

(viii) C^n is the greatest l-ideal that satisfies (iii).

In the proof of Theorem 6.1 some of these properties are verified.

THEOREM 6.1. *An l-group is a small lexico-sum of o-groups if and only if L satisfies* (F): *Each* $0<a\in L$ *is greater than at most a finite number of disjoint elements.*

Note that Theorem 2.2 is a special case of this theorem. We could make use of Theorem 2.2 and shorten the following proof slightly, but we prefer to give an independent proof. As remarked above, if L is a small lexico-sum of o-groups, then L satisfies (F). We prove the converse. *In all that follows assume that L satisfies* (F). By Theorem 5.2, L has a basis. Let $[A]=\sum\boldsymbol{+}A_\gamma$ be the basis group of L. $[A]$ will serve for C^1 and the main body of the proof is concerned with the inductive procedure of getting from C^n to C^{n+1}.

Suppose that $\{B_\gamma:\gamma\in\Gamma\}$ is a set of convex subgroups of L that satisfy the following four conditions.

(a) Each B_γ is a proper lexico-extension of an l-ideal I_γ of B_γ.

(b) $B=[\bigcup_{\gamma\in\Gamma}B_\gamma]=\sum_{\gamma\in\Gamma}\boldsymbol{+}B_\gamma$ and B is an l-ideal of L.

(c) No strictly positive element of L is disjoint from every B_γ.

(d) $B_\gamma^+ = \{x \in L^+ : x \cap B_\delta^+ = 0$ for all $\delta \neq \gamma$ in $\Gamma\}$ for each $\gamma \in \Gamma$.

We next prove that

(e) B^+ is the set of all elements of L^+ which do not exceed any B_γ.

Proof. Clearly no element of B^+ exceeds any B_γ. Conversely let c be an element of L^+ not exceeding any B_γ. By Lemma 6.2, $c \in B_\gamma^+ + B_\gamma^*$ for each $\gamma \in \Gamma$. Hence $c = c_\gamma + c_\gamma^*$ with c_γ in B_γ^+ and $c_\gamma^* \cap B_\gamma^+ = 0$. Since L satisfies (F), $c_\gamma > 0$ for only a finite subset $\{\gamma_1, \cdots, \gamma_n\}$ of Γ. Since c exceeds the mutually disjoint elements $c_{\gamma_1}, \cdots, c_{\gamma_n}$, it exceeds their sum, and hence $c = c_{\gamma_1} + \cdots + c_{\gamma_n} + a$ with $a \geqq 0$. Since $a \leqq c$, $a \cap B_\gamma^+ = 0$ for all γ in $\Gamma \backslash \{\gamma_1, \cdots, \gamma_n\}$. From $c = c_{\gamma_i} + c_{\gamma_i}^* = c_{\gamma_1} + \cdots + c_{\gamma_n} + a$ we conclude that $a \leqq c_{\gamma_i}^*$ (using the fact that the c_{γ_i} commute with each other) and so $a \cap B_{\gamma_i} = 0$ for $i = 1, \cdots, n$. So by (c), $a = 0$ and we conclude that $c \in B_{\gamma_1}^+ + \cdots + B_{\gamma_n}^+ \subseteq B^+$.

Let us call an element d of L^+ *dominating* with respect to $\{B_\gamma : \gamma \in \Gamma\}$ if, for each γ in Γ, either $d > B_\gamma^+$ or $d \cap B_\gamma^+ = 0$.

(f) If $a \in L^+ \backslash B^+$, then $a = b + d$, where $b \in B^+$ and d is a dominating element such that $d > B_\gamma$ if and only if $a > B_\gamma$.

Proof. By (F), a is disjoint from all but a finite number of the B_γ. By (e), a must exceed at least one B_γ. Hence there exist two finite disjoint subsets $\{\alpha_1, \cdots, \alpha_m\}$ and $\{\beta_1, \cdots, \beta_n\}$ of Γ, the former not empty, such that $a > B_{\alpha_i}$ $(i = 1, \cdots, m)$; $a \not> B_{\beta_j}$ and $a \cap B_{\beta_j} \neq 0$ $(j = 1, \cdots, n)$; and $a \cap B_\gamma^+ = 0$ for all other elements γ of Γ. If $\{\beta_1, \cdots, \beta_n\} = \square$, then a itself is dominating and (f) holds with $b = 0$. Assume that $\{\beta_1, \cdots, \beta_n\} \neq \square$. By Lemma 6.2, $a = b_j + b_j^*$ with $b_j \in B_{\beta_j}$ and $b_j^* \cap B_{\beta_j} = 0$ $(j = 1, \cdots, n)$. Let $b = b_1 + \cdots + b_n$. Since $b_1, \cdots, b_n$ are disjoint and each $b_j \leqq a$, it follows that $b \leqq a$. Thus $a = b + d$ with $d \geqq 0$. Furthermore, $d > 0$ since otherwise $a \in B^+$. If $a \cap B_\gamma^+ = 0$, then $d \cap B_\gamma^+ = 0$ because $0 < d \leqq a$. From $a = b_j + b_j^* = b + d = b_1 + \cdots + b_n + d$ it follows that $d \leqq b_j^*$ and so $d \cap B_{\beta_j}^+ = 0$ $(j = 1, \cdots, n)$. If $x \in B_{\alpha_i}^+$, then $x < a = b + d$, and since $x \cap b = 0$ it follows that $x \leqq d$. Hence d is a dominating element and dominates the same B_γ as a—namely, $B_{\alpha_1}, \cdots, B_{\alpha_m}$.

Next we define a relation $\sim$ on Γ as follows: $\gamma \sim \delta (\gamma, \delta \in \Gamma)$ if $\gamma = \delta$ or if $\gamma \neq \delta$ and

(i) there exists an x in L^+ such that $x > B_\gamma$ and $x > B_\delta$, and

(ii) for each x in L^+, $x > B_\gamma$ if and only if $x > B_\delta$.

Clearly $\sim$ is an equivalence relation, and by (F) each equivalence class is a finite subset of Γ. Let $\{\alpha_1, \cdots, \alpha_m\}$ be an equivalence class with $m > 1$. Let D' be the set of all elements d of L^+ such that $d > B_{\alpha_i}$ $(i = 1, \cdots, m)$ and $d \cap B_\gamma = 0$ for all other γ in Γ. $D' \neq \square$, for by the definition of $\sim$, it follows that there exists an x in L^+ such that $x > B_{\alpha_i}$ $(i = 1, \cdots, m)$ and $x \not> B_\gamma$ for all other γ in Γ, and by (f), $x = b + d$, where $b \in B^+$ and $d \in D'$. Clearly D' is a convex subsemigroup of L^+.

Let $B' = B_{\alpha_1}^+ + \cdots + B_{\alpha_m}^+$, and let $D = B' \cup D'$. Since $B' + D' \subseteq D'$ and $D' + B' \subseteq D'$, D is a subsemigroup of L^+ that contains zero. Suppose that

$u<x<v$ for $u, v\in D$. We wish to show that $x\in D$. This is clear if $v\in B'$. If $v\in D'$, then $x\cap B_\gamma^+=0$ for all γ in Γ except $\alpha_1, \cdots, \alpha_m$. If $x>B_{\alpha_i}$ for some i, then $x\in D'$. Suppose that $x\notin D'$. Then $x\not> B_{\alpha_i}$ for $i=1, \cdots, m$, and hence $x \not> B_\gamma$ for all $\gamma \in \Gamma$. Therefore by (e), $x \in B^+$ and since $x \cap B_\gamma^+ = 0$ for all $\gamma\in\Gamma\setminus\{\alpha_1, \cdots, \alpha_m\}$, it follows that $x\in B'$. Therefore D is a convex subsemigroup of L^+ that contains 0. By Theorem 2.1, $[D]=\{a-b: a, b\in D\}$. Let $b_1\in B_{\alpha_1}^+$, $b_2\in B_{\alpha_2}^+$ and $d\in D'$. Since $b_1\cap b_2=0$, $d+b_1-d\cap d+b_2-d=0$. Thus $d+b_1-d\in B'$ and it follows that $[B']$ is normal in $[D]$. Therefore $[D]$ is a convex subgroup of L which is a nontrivial lexico-extension of $[B']$.

The foregoing applies to each equivalence class containing more than one element. For an equivalence class consisting of one element γ of Γ it follows from (d) (and this is the first use made of (d)) that $D'=\square$. In which case we take $[D]$ to be B_γ. Now construct the subgroups one to each equivalence class mod $\sim$, as described above and denote them by $\{D_\lambda: \lambda\in\Lambda\}$.

Proposition 6.1. $\{D_\lambda: \lambda\in\Lambda\}$ *also has the properties* (a), (b), (c) *and* (d) *and, for each* λ *in* Λ, *either* D_λ *is a nontrivial lexico-extension of a finite cardinal sum of the* B_γ *or else* D_λ *is equal to one of the* B_γ.

Proof. (a) has already been established. As for (b) it is clear that if $\lambda\neq\mu$ in Λ, then $D_\lambda^+\cap D_\mu^+=0$, and so by Theorem 2.1, the group D that is generated by all the D_λ is the small cardinal sum of the D_λ. Now by (b), $x+B-x=B$ for all x in L. Since each B_γ is a nontrivial lexico-extension, it is cardinally indecomposable, and the representation of B as a small cardinal sum of cardinally indecomposable convex subgroups is unique [2, p. 222]. It follows that $x+B_\gamma-x=B_\alpha$ for some $\alpha\in\Gamma$.

Suppose that D_λ is a lexico-extension of $B_{\alpha_1}, \cdots, B_{\alpha_m}$, where $\{\alpha_1, \cdots, \alpha_m\}$ is an equivalence class mod $\sim$. Let $B_{\beta_i}=x+B_{\alpha_i}-x$. Then $\{\beta_1, \cdots, \beta_m\}$ is an equivalence class, and $x+D_\lambda-x$ is easily seen to be the lexico-extension D_μ of $B_{\beta_1}+\cdots+B_{\beta_m}$ constructed above. Thus the inner automorphisms just permute the components D_λ of D and so D is an l-ideal. Therefore $\{D_\lambda: \lambda\in\Lambda\}$ satisfies (b).

(c) If $0<a\in L$, then $a\cap B_\gamma\neq 0$ for some γ in Γ, since $\{B_\gamma: \gamma\in\Gamma\}$ satisfies (c). Thus $a\cap b_\gamma\neq 0$ for some $0<b_\gamma\in B_\gamma$. But $B_\gamma\subseteq D_\lambda$ for some λ, and hence $\{D_\lambda: \lambda\in\Lambda\}$ satisfies (c).

(d) Let $\lambda\in\Lambda$ and let a be an element of L^+ which is disjoint from every D_μ^+ with $\mu\neq\lambda$. Let D_λ be the lexico-extension of $B_{\alpha_1}\boldsymbol{+}\cdots\boldsymbol{+}B_{\alpha_m}$ constructed above, $\{\alpha_1, \cdots, \alpha_m\}$ being an equivalence class mod $\sim$. If $a\in B$, then $a\in B_{\alpha_1}\boldsymbol{+}\cdots\boldsymbol{+}B_{\alpha_m}\subseteq D_\lambda$ because a is disjoint from every D_μ with $\mu\neq\lambda$, hence from every B_γ with γ not in $\{\alpha_1, \cdots, \alpha_m\}$. If $a\notin B$, then by (f), $a=b+d$ with $b\in B^+$ and d a dominating element. But d is disjoint from every B_γ with γ not in $\{\alpha_1, \cdots, \alpha_m\}$. Thus it is clear that d must dominate $B_{\alpha_1}, \cdots, B_{\alpha_m}$: thus $d\in D_\lambda$ and so $a\in D_\lambda$.

Therefore the induction procedure for constructing C^{n+1} from C^n is now clear, and all that remains is to show that $\bigcup_{i=1}^{\infty} C^n = L$. Suppose (by way of contradiction) that $d \in L^+ \setminus \bigcup_{i=1}^{\infty} C^n$, and that d has a minimal length m in the following sense. There exists a positive integer m such that d is disjoint from all but m of the components C_γ^n of C^n, and for no other d in $L^+ \setminus \bigcup_{i=1}^{\infty} C^n$ or choice of n can this number m be reduced. It follows from (d) that $m > 1$. Write B for C^n and let $B_{\alpha_1}, \cdots, B_{\alpha_m}$ be the m components C_γ^n of B from which d is not disjoint. By (f) and the minimality of m, it follows that d is a dominating element and that it dominates $B_{\alpha_1}, \cdots, B_{\alpha_m}$. If two distinct α_i were equivalent, then d would be disjoint from all but $r < m$ of the subgroups C_α^{n+1}, contrary to the minimality of m. Hence all the α_i are inequivalent and there exists a $c \in L^+$ such that $c > B_{\alpha_i}$ for some i and $c \not> B_{\alpha_j}$ for some $j \neq i$. We may assume (using (e) and (f)) that $c > B_{\alpha_i}$ and $c \cap B_{\alpha_j} = 0$. Thus $c \cap d > B_{\alpha_i}$, $(c \cap d) \cap B_{\alpha_j} = 0$, and $(c \cap d) \cap C_\gamma^n = 0$ for all γ not in $\{\alpha_1, \cdots, \alpha_m\}$, but this contradicts the minimality of m. Therefore $\bigcup_{i=1}^{\infty} C^n = L$, and at long last this completes the proof of Theorem 6.1.

Another way to prove Theorem 6.1 is by use of the following lemma (the proof of which is quite short).

LEMMA 6.3. *Suppose that L satisfies* (F) *and that B is an l-ideal of L that contains the basis group $[A]$ of L. The L/B satisfies* (F), *and hence L/B has a basis. Moreover, either $L = B$ or there exists an l-ideal C of L such that $C \supseteq B$ and C/B is the basis group of L/B.*

Using this lemma we can construct a chain

$$[A] = A^1 \subseteq A^2 \subseteq A^3 \subseteq \cdots$$

of l-ideals of L such that L/A^n satisfies (F) and A^{n+1}/A^n is the basis group of L/A^n. Then by arguments similar to those used in the given proof of Theorem 6.1 one shows that $\bigcup_{n=1}^{\infty} A^n = L$ and that the A^n satisfy the definition for a small lexico-sum. Proposition (f) is the key to the given proof, and the fact that L/B satisfies (F) is the key to the alternate proof. The next theorem shows that each l-group with a basis contains a greatest l-ideal that satisfies (F). The structure of this ideal is then given by Theorem 6.1.

THEOREM 6.2. *Suppose that L has a basis and let $F = \{0 \leqq x \in L : x$ exceeds at most a finite number of disjoint elements$\}$. Then $[F]$ is an l-ideal of L that contains the basis group $[A] = \sum + A_\gamma$ of L and $[F]$ satisfies condition* (F). *Thus $[F]$ is a small lexico-sum of the A_γ. If T is any other convex subgroup of L that satisfies* (F), *then $T \subseteq [F]$. If $0 < a \in L \setminus [F]$, then there exist an infinite number of $\gamma \in \Gamma$ such that $a > a_\gamma$ for some $0 < a_\gamma \in A_\gamma$.*

Proof. Clearly $F \supseteq \sum + A_\gamma^+$ and F is convex. Consider $x, y \in F$. $x \cap A_\gamma^+ = y \cap A_\gamma^+ = 0$ for all but a finite number of γ. Therefore $(x+y) \cap A_\gamma^+ = 0$ for all but a finite number of γ. It follows from Lemma 2.1 that $(L^{x+y})^+ \cap A_\gamma^+ = 0$ for all but a finite number of γ, and so L^{x+y} has a finite basis. Thus $x+y$ is greater than at most a finite number of disjoint elements in L^{x+y} and hence in L. Therefore F is a convex subsemigroup of L and clearly F is normal. It follows from Theorem 2.1 that $[F] = \{a-b: a, b \in F\}$ and that $[F]$ is an l-ideal of L that contains $[A]$ and satisfies (F). If $0 < a \in L \backslash [F]$ and $a \cap A_\gamma^+ = 0$ for all but a finite number n of the γ, then it follows that a is greater than at most n disjoint elements, and hence $a \in F$, a contradiction. Therefore $a \cap A_\gamma^+ \neq 0$ for an infinite number of γ.

The *interval topology* of L is defined by taking as a sub-basis for the closed sets all closed infinite intervals $[a, \infty]$ and $[-\infty, a]$. Birkhoff [2] has asked the question: Is L a topological group in its interval topology? Northam [6] has shown that the answer is no for the l-group in Example I of §10, and Choe [3] has shown that the answer is no for any noncyclic l-group that satisfies the chain condition (see the corollary to Theorem 7.2). We shall show that the answer is no for all nonordered l-groups that satisfy (F). Note that if L is ordered, then the open intervals $a \leqq x \leqq b$ are open sets in the interval topology, and it follows that L is a topological group.

LEMMA 6.4. *The interval topology of L is Hausdorff if and only if given any two distinct points a and b there is a covering of L by means of a finite number of closed infinite intervals such that no interval in this covering contains both a and b.*

This follows immediately from the definitions of a Hausdorff space and a sub-basis (or see [6, Proposition I]).

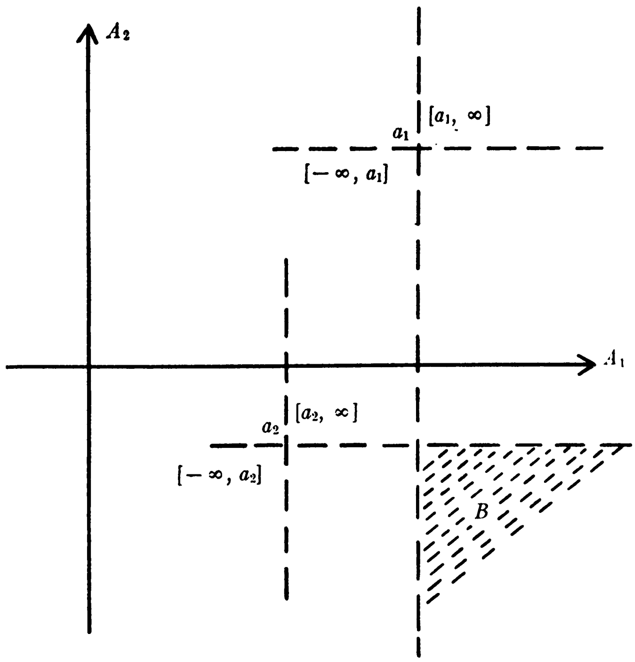
COROLLARY. *If L is Hausdorff in its interval topology, then*

$$(*) \qquad L = \bigcup_{i=1}^{n} ([-\infty, a_i] \cup [a_i, \infty])$$

for some finite subset $a_1, \cdots, a_n$ *of L.*

LEMMA 6.5. *If* $L = A_1 \dotplus A_2$, *where* A_1 *and* A_2 *are nonzero o-groups, then L does not satisfy* (*) *and hence L is not a topological group in its interval topology.*

Proof. Suppose (by way of contradiction) that L is a topological group in its interval topology. Then since the interval topology is T_1 and a topological group is regular, L is Hausdorff. Thus there exists a finite subset $A = a_1, \cdots, a_n$ of L that satisfies (*). $a_i = (a_{i1}, a_{i2})$, where $a_{i1} \in A_1$ and $a_{i2} \in A_2$ for $i = 1, \cdots, n$. We may assume without loss of generality that a_1 is the element with smallest second coordinate in the subset of all elements in A with largest first coordinate, and that a_2 is the element with the largest first coordinate in the subset of all elements in A with the smallest second coordinate. Then we have the following "picture" of L.



It follows that the elements in B are not contained in

$$\bigcup_{i=1}^{n} ([-\infty, a_i] \cup [a_i, \infty]), \text{ a contradiction.}$$

THEOREM 6.3. *If L satisfies* (F), *then L is a topological group in its interval topology if and only if L is ordered.*

Proof. By the above remarks each o-group is a topological group in its interval topology. Suppose that L satisfies (F) and L is not ordered. It follows from our structure Theorem 6.1 that L contains a convex subgroup of the form $A_1 \dotplus A_2$ that satisfies

(i) A_1 and A_2 are maximal ordered convex subgroups, and

(ii) $x > A_1$ if and only if $x > A_2$ for all $x \in L$.

Choose $a = (0, 0)$ and $b = (a_1, a_2)$ in $A_1 \dotplus A_2$, where $a_1 > 0$ and $a_2 > 0$. Assume (by way of contradiction) that L is a topological group in its interval topology. By Lemma 6.4 we have $b_1, \cdots, b_m, c_1, \cdots, c_n$ in L such that

$$L = \left(\bigcup_{i=1}^{m} [-\infty, b_i] \right) \cup \left(\bigcup_{i=1}^{n} [c_i, \infty] \right)$$

and none of these infinite closed intervals contain both a and b. Let

$$B_i = [-\infty, b_i] \cap (A_1 \dotplus A_2) = \{ x \in A_1 \dotplus A_2 : x \leqq b_i \},$$

$$A_1(b_i) = \{x \in A_1 : x + y \leqq b_i \text{ for some } y \in A_2\},$$
$$A_2(b_i) = \{y \in A_2 : x + y \leqq b_i \text{ for some } x \in A_1\}$$

for $i=1, \cdots, m$. If $A_1(b_i)=A_1$, then by the structure of L it follows that $b_i > A_1$ and hence by (ii) $b_i > A_1 \boldsymbol{+} A_2$. But then $[-\infty, b_i]$ contains both a and b, a contradiction. Thus if $B_i \neq \square$, then $A_1(b_i)$ and $A_2(b_i)$ are proper lower segments of A_1 and A_2 respectively. Pick $b_{i1} \in A_1 \backslash A_1(b_i)$ and $b_{i2} \in A_2 \backslash A_2(b_i)$. Then $[-\infty, b_{i1}+b_{i2}] \cap (A_1+A_2) \supseteq B_i$. Dually, for each $C_i = [c_i, \infty] \cap (A_1 \boldsymbol{+} A_2)$, there exist $c_{i1} \in A_1$ and $c_{i2} \in A_2$ such that $[c_{i1}+c_{i2}, \infty] \cap (A_1+A_2) \supseteq C_i$. But

$$A_1 \boldsymbol{+} A_2 = \left(\bigcup_{i=1}^{m} B_i\right) \cup \left(\bigcup_{i=1}^{n} C_i\right).$$

It follows that $A_1 \boldsymbol{+} A_2$ satisfies condition (*) in the corollary to Lemma 6.4, but this is impossible by Lemma 6.5. Therefore L is not a topological group in its interval topology.

The problem now is to find an example of a nonordered l-group that is a topological group in its interval topology.

7. Ordered convex subgroups of L that are not bounded from above. Clearly an ordered convex subgroup of L that is not bounded from above is also not bounded from below. Let $\{A_\delta : \delta \in \Delta\}$ be the set of all ordered convex subgroups of L each of which is not bounded from above. Each A_δ is a convex maximal chain of elements in L, and conversely by Theorem 3.1, each maximal chain of L that is convex and contains 0 is one of the A_δ.

LEMMA 7.1. *For each $\delta \in \Delta$, $L = A_\delta \boldsymbol{+} A^\delta$, where A^δ is the subgroup of L that is generated by $\{x \in L^+ : x \cap A_\delta^+ = 0\}$. A^δ is uniquely determined by A_δ.*

Proof. By Lemma 6.2, $L \supseteq A_\delta \boldsymbol{+} A^\delta$. Pick $0 < a \in L$. $a \not> A_\delta$. Thus either $a \in A^\delta$ or there exists $0 < c \in A_\delta$ such that $0 < a \cap c < c$. In the latter case $a = a \cap c + a'$, $c = a \cap c + c'$ and $a' \cap c' = 0$. Thus $a = a'$ mod A_δ and $a' \in A^\delta$. Therefore $a \in A_\delta + A^\delta$ and hence $A = A_\delta \boldsymbol{+} A^\delta$. It follows from Lemma 6.1 that A^δ is uniquely determined by A_δ.

Now let $D = \bigcap_{\delta \in \Delta} A^{\delta +}$. Then since $A^{\delta +} = \{x \in L^+ : x \cap A_\delta^+ = 0\}$,

$$D = \{x \in L^+ : x \cap A_\delta^+ = 0 \text{ for all } \delta \text{ in } \Delta\}.$$

$[D] = \bigcap_{\delta \in \Delta} A^\delta$. For $\bigcap_{\delta \in \Delta} A^\delta$ is a convex subgroup of L that contains D, and hence $[D] \subseteq \bigcap A^\delta$. If $x \in \bigcap A^\delta$, then $x = a - b$, where a and b are positive elements in $\bigcap A^\delta$. Thus $a, b \in D = \bigcap A^{\delta +}$, and hence $x = a - b$ belongs to $[D]$.

Let $D^* = \{x \in L^+ : x \cap D = 0\}$. Then $[D^*] \supseteq \sum_{\delta \in \Delta} \boldsymbol{+} A_\delta$, and by Lemma 6.2

$$L \supseteq [D^*] \boldsymbol{+} [D].$$

Consider $x \in [D^*]$ and $\delta \in \Delta$. $x = x_\delta + x^\delta$ with $x_\delta \in A_\delta$ and $x^\delta \in A^\delta$. Define that $x\sigma = (\cdots, x_\delta, \cdots)$.

LEMMA 7.2. *σ is an l-isomorphism of $[D^*]$ onto a sublattice of the large cardinal sum V of the A_δ that contains the small cardinal sum U of the A_δ.*

Proof. Clearly σ is a group homomorphism of $[D^*]$ into V. If $x\in[D^*]$ and $x\sigma=\theta$ (the zero of V), then $x_\delta=0$ for all $\delta\in\Delta$, and hence $x\in\bigcap A^\delta=[D]$. Thus $x\in[D^*]\cap[D]=0$ and so σ is an isomorphism. If $0<x\in A_\delta$ and $\delta\neq\gamma\in\Delta$, then $x=x_\gamma+x^\gamma=x_\gamma\cup x^\gamma$ with $x_\gamma\in A_\gamma$ and $x^\gamma\in A^\gamma$. Thus $x_\gamma=x\cap x_\gamma \in A_\delta\cap A_\gamma=0$, and $(x\sigma)_\gamma=x_\gamma=0$ and $(x\sigma)_\delta=x$. It follows that $[D^*]\sigma\supseteq U$. To prove that $[D^*]\sigma$ is a sublattice of V it suffices to show that $x'\cup\theta\in[D^*]\sigma$ for all x' in $[D^*]\sigma$ [2, p. 215]. $x'\cup\theta = x\sigma\cup\theta = (\cdots, x_\delta, \cdots)\cup\theta = (\cdots, x_\delta\cup 0, \cdots)$, where $x=x'\sigma^{-1}$. For each $\delta\in\Delta$, $x=x_\delta+x^\delta$ and $x\cup 0 = (x_\delta\cup 0)+(x^\delta\cup 0)$. Therefore $(x\cup 0)\sigma=(\cdots, x_\delta\cup 0, \cdots)=x\sigma\cup\theta$. Since $(x\cup 0)\sigma=x\sigma\cup\theta$ for all $x\in[D^*]$, it follows that σ and σ^{-1} preserve order.

COROLLARY. *For each δ in Δ pick an $a_\delta(0<a_\delta\in A_\delta)$. Then $\{a_\delta: \delta\in\Delta\}$ is a basis for $[D^*]$.*

For clearly the set of all $(0, \cdots, 0, a_\delta, 0, \cdots, 0)$ is a basis for V and hence for $[D^*]\sigma$.

THEOREM 7.1. *If L satisfies* (F) *or if Δ is finite, then*

$$L = \left(\sum_{\delta\in\Delta} + A_\delta\right) + [D].$$

Proof. If Δ is finite, then the conclusion is an immediate consequence of Lemma 7.1, and if L satisfies (F), then the conclusion follows from Theorem 6.1.

COROLLARY. *L is a small cardinal sum of o-groups if and only if L satisfies* (F) *and no maximal convex ordered subgroup of L is bounded from above.*

Proof. If L satisfies the last two conditions, then $[D]=0$, and hence $L=\sum + A_\delta$. Thus in this case L coincides with its basis group. The converse is obvious.

If A is an l-group, then we say that A can be *embedded* in a large cardinal sum V of o-groups V_δ ($\delta\in\Delta$) if A is l-isomorphic to a sublattice of V that contains the small cardinal sum U of the V. For example, Lemma 7.2 states that $[D^*]$ can be embedded in the large cardinal sum of the o-groups A_δ. Actually, if the image S of A contains the small direct sum of the V_δ and S is a lattice with respect to the induced partial order, then S is necessarily a sublattice of V.

THEOREM 7.2. *The following are equivalent.*

(i) *L has a basis and no maximal convex ordered subgroup of L is bounded from above.*

(ii) *L can be embedded in a large cardinal sum of o-groups.*

Proof. If L satisfies (i), then $D=0$ and hence $[D^*]=L$. Thus by Lemma 7.2, L satisfies (ii). The converse is obvious.

COROLLARY. *If L satisfies the chain condition*

(C) *every nonvoid subset of L^+ contains a minimal element,*

then $L=\sum+C_\gamma$, where the C_γ are infinite cyclic groups.

Proof. If $0<x\in L$, then there exists a $y\in L$ such that $0<y\leqq x$ and y covers 0. Thus y is basic, and hence by Theorem 5.1, L has a basis. Let $[A]=\sum+A_\gamma$ be the basis group of L. The only ordered group that satisfies (C) is the infinite cyclic group. Thus the A_γ are infinite cyclic groups. Since L satisfies (C), the A_γ are not bounded from above. Thus L satisfies condition (i) of our theorem, and (C) assures that L is the small direct sum $\sum+A_\gamma$. This well known result is also an immediate consequence of Theorem 6.1.

L is *Archimedean* if for all a, b in L

$$na \leqq b \text{ for } n = 1, 2, \cdots \text{ implies that } a \leqq 0.$$

It is well known that an Archimedean o-group is o-isomorphic to a subgroup of the (naturally ordered) additive group R of real numbers.

THEOREM 7.3. *The following are equivalent.*

(i) *L is Archimedean and has a basis.*

(ii) *L can be embedded in a large cardinal sum of subgroups of R. In particular, L is a small cardinal sum of Archimedean o-groups if and only if L is Archimedean and satisfies* (F).

Proof. Suppose that L satisfies (i) and let $[A]=\sum+A_\gamma$ be the basis group of L. It follows that the A_γ are Archimedean o-groups. Clearly the A_γ are not bounded from above. Thus (ii) follows from Theorem 7.2. The converse is obvious and the last statement follows from the corollary to Theorem 7.1.

COROLLARY I. *Suppose that L is σ-complete and has a basis. Let $[A]=\sum+A_\gamma$ be the basis group of L. Then each A_γ is o-isomorphic to R or to the group I of integers, and L can be embedded into the large cardinal sum of the A_γ. In particular, L is a small cardinal sum of o-groups each of which is o-isomorphic to R or to I if and only if L is σ-complete and L satisfies* (F).

Proof. L is Archimedean and the A_γ are σ-complete o-groups. Thus each A_γ is o-isomorphic to R or to I.

COROLLARY II. *Suppose that L is complete and has a basis. Then L is l-isomorphic to the large cardinal sum of the A_γ if and only if every basis of L has an upper bound.*

Proof. Suppose that every basis of L has an upper bound. Let $S=\{a_\gamma:\gamma\in\Gamma\}$ be a basis for L and let $T=\{d_\gamma:\gamma\in\Gamma\}$, where for each γ in Γ either $d_\gamma=0$ or

$d_\gamma = a_\gamma$. Then T has an upper bound, and hence a least upper bound q in L. Let σ be the l-isomorphism given in Lemma 7.2. Thus σ is an embedding of L into the large cardinal sum V of the A_γ. $q\sigma = (\cdots, q_\gamma, \cdots)$ and $q_\gamma = d_\gamma$, for otherwise q is not a least upper bound for T. Thus $L\sigma$ contains all the positive elements of V and hence $L\sigma = V$. The converse is obvious.

Example of an abelian l-group L with a basis such that $L \neq [D^*] \boldsymbol{+} [D]$. Let I_i be the group of integers ($i=1, 2, \cdots$) and $U(V)$ be the small (large) cardinal sum of the I_i. Let $W = P \oplus Q$, where $P = Q =$ integers, and $(p, q) \in P + Q$ is positive if $q > 0$ or $q = 0$ and $p > 0$. Finally let L be the subgroup of $V \boldsymbol{+} W$ that is generated by U, P and the element $((1, 1, 1, \cdots), (0, 1))$. Then it is easy to show that L is a sublattice of $V \boldsymbol{+} W$, that $[D] = P$ and that $[D^*] = U$.

Example of an Archimedean l-group L such that $L \neq [D^*] \boldsymbol{+} [D]$. Let U and V be the same as in the last example, and let W be the l-group of all continuous real valued functions defined on the closed unit interval (see Example I, §10). Let $W' = \{f \in W : f(1/2) = 0\}$. Let L be the subgroup of $V \boldsymbol{+} W$ that is generated by U, W' and the element $((1, 1, 1, \cdots), g)$, where $g(x) = 1$ for all x in $[0, 1]$. Then $[W' \cup \{g\}] = \{f \in W : f(1/2) = \text{an integer}\}$ and this is a sublattice of W. It is easy to show that L is a sublattice of $V \boldsymbol{+} W$, that $[D] = W'$ and that $[D^*] = U$.

8. **Attempts to generalize Theorem** 6.1. What is needed is a structure theorem for l-groups with a basis. The proof of Theorem 6.1 depends upon the fact that if L satisfies (F), then so does $L/[A]$, where $[A]$ is the basis group of L. In particular, $L/[A]$ has a basis. The following example shows that if L has a basis, then $L/[A]$ need not have a basis.

For each positive integer n let I_n be the group of integers, and let $U(V)$ be the small (large) cardinal sum of the I_n. $(1, 0, 0, \cdots), (0, 1, 0, \cdots), \cdots$ is a basis for V and U is the basis group of V. It is easy to show that each positive element in $G = V/U$ is greater than two disjoint elements, and hence G does not have a basis. In fact, G has no convex ordered subgroup except the zero subgroup. For example $(1, 1, 1, \cdots) + U$ is greater than $(1, 0, 1, 0, \cdots) + U$ and $(0, 1, 0, 1, \cdots) + U$. Moreover $(1, 1, 1, \cdots) \in V \backslash U$, but $(1, 1, 1, \cdots)$ is not greater than

$$A_n = \{(x_1, x_2, \cdots) \in V : x_i = 0 \text{ for all } i \neq n\}.$$

This shows that the hypothesis in Proposition 4.5 cannot be removed.

Let $S = \{a_\gamma : \gamma \in \Gamma\}$ be a basis for L and let $[A] = \sum \boldsymbol{+} A_\gamma$ be the basis group of L. Define that

$$\begin{aligned} H &= \{x + L^+ : x \not> A_\gamma \text{ for each } \gamma \in \Gamma\} \\ &= \{x \in L^+ : \text{for each } \gamma \in \Gamma \text{ there exists } 0 < c \in A_\gamma \quad \text{for which } x \cap c < c\}. \end{aligned}$$

LEMMA 8.1. *H is a convex subsemigroup of L that contains* $[A]^+$. *H is invariant with respect to the l-automorphisms of L.*

Proof. If $0<y<x$ and $x\in H$, then for each γ in Γ, there exists a strictly positive element c in A_γ such that $y\cap c\leqq x\cap c<c$. Therefore $y\in H$ and hence H is convex. Consider $x,\ y\in H$. There exist strictly positive elements a and b in A_γ such that $x\cap a<a$ and $y\cap b<b$. Let $d=a+b$. Then $x\cap d=x\cap a$ and $y\cap d=y\cap b$, and hence $(x+y)\cap d\leqq(x\cap d)+(y\cap d)=(x\cap a)+(y\cap b)<a+b=d$. Therefore $x+y\in H$, and hence H is a semigroup. Clearly $H\supset[A]^+$.

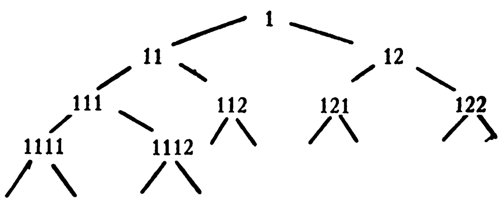
Finally let π be an l-automorphism of L. We wish to show that $H\pi\subseteq H$. Consider $0<h\in H$. For each γ in Γ there exists a strictly positive element h_γ in A_γ such that $h\cap h_\gamma<h$. Thus $h\pi\cap h_\gamma\pi\leqq h_\gamma\pi$. Now π permutes the A_γ^+ (see Theorem 5.3). Thus there is one and only one $h_\gamma\pi$ in each A_γ. Therefore $h\pi\in H$.

THEOREM 8.1. *$[H]$ is an l-ideal of L and $[H]$ can be embedded in the large cardinal sum of the o-groups A_γ.*

Proof. Since H is a convex subsemigroup of L^+ that contains 0, $[H]=\{a-b\colon a,\ b\in H\}$ and $[H]$ is a convex subgroup of L (by Theorem 2.1). By Lemma 8.1, H is normal and hence $[H]$ is an l-ideal of L. The A_γ are the maximal convex o-subgroups of $[H]$ and none is bounded from above in $[H]$. Thus the last part of the theorem is an immediate consequence of Theorem 7.2.

Note that if $0<a\in L\backslash[H]$, then $a>A_\gamma$ for some γ in Γ. If $L/[H]$ has a basis, then we can continue this process. Unfortunately, the next example shows that $L/[H]$ need not have a basis.

Let S be the set of all finite sequences of 1's and 2's each of which starts with a 1. Partially order S as in the following diagram.



Let N be the trivially ordered set of all positive integers greater than or equal to 3. Let $T=S\cup N$. We extend the partial orders of S and N to a partial ordering of T as follows

$$1>3,\ 4,\ 5,\cdots$$

$$11>3,\ 5,\ 7,\cdots$$

$$12>4,\ 6,\ 8,\cdots$$

$$111>3,\ 7,\ 11,\cdots$$

$$112>5,\ 9,\ 13,\cdots.$$

For example $1121 > d_1, d_2, d_3, \cdots$, where the sequence $\{d_i\}$ is obtained from $3, 4, 5, \cdots$ by

1st taking the odd terms of $3, 4, 5, \cdots$ to get a sequence $\{a_i\}$.

2nd taking the odd terms of $\{a_i\}$ to get a sequence $\{b_i\}$.

3rd taking the even terms of $\{b_i\}$ to get a sequence $\{c_i\}$.

4th taking the odd terms of $\{c_i\}$ to get the sequence $\{d_i\}$.

For each t in T let I_t be the group of integers, and let L be the large direct sum of the I_t. For each $a = (\cdots, a_t, \cdots)$ in L let F_a be the set of all the nonzero components a_t of a each of which has a subscript t that is maximal with respect to the set of all subscripts of the nonzero components of a. Define that a is positive if each element in F_a is positive. It follows that L is an l-group with a basis, and that the basis group of L is the small direct sum of the I_n $(n \in N)$. $[H]$ is the large direct sum of the I_n $(n \in N)$, and $L/[H]$ is l-isomorphic to the large direct sum B of the I_s $(s \in S)$. B has no basis. In fact, B has no convex ordered subgroups except $\{0\}$.

9. **The subgroup of L that is generated by the nonunits.** We shall call an element u in L a *nonunit* if $u > 0$ and $u \cap v = 0$ for some $0 < v \in L$ (in Birkhoff's terminology u is not a weak unit).

LEMMA 9.1. *Let H be an l-ideal of L. Then the following are equivalent.*

(1) $L = \langle H \rangle$.

(2) *H contains all the nonunits of L.*

(3) *Each nonzero coset in L/H consists entirely of positive elements or entirely of negative elements.*

Proof. Suppose that $L = \langle H \rangle$, and assume (by way of contradiction) that a nonunit a belongs to $L \backslash H$. Then there exists $0 < b \in L$ such that $a \cap b = 0$. If $b \in H$, then $a > b$, a contradiction. If $b \in L \backslash H$, then $H = H + (a \cap b) = (H + a) \cap (H + b)$, but this is impossible because L/H is ordered. Therefore (1) implies (2). Next assume that (2) is satisfied, and let A be a nonzero element in L/H. Then $A = a + H$, where $a \in L \backslash H$. $a = a^+ - a^-$, where $a^+ = a \cup 0$, $a^- = -a \cup 0 = -(a \cap 0)$ and $a^+ \cap a^- = 0$. If a^+ and a^- are both nonzero, then they are both nonunits, and hence $a \in H$. Thus either $a^+ = 0$ or $a^- = 0$, and hence either $a < 0$ or $a > 0$. Therefore each element in A is either positive or negative, and since the natural homomorphism of L onto L/H preserves order, (3) is satisfied. Finally suppose that (3) is satisfied. Then clearly L/H is an o-group. Consider $0 < a \in L \backslash H$. Since each element in $a + H$ is positive, it follows that a exceeds every element in H. Therefore $L = \langle H \rangle$.

LEMMA 9.2. *Let N be the set of all nonunits of L.*

(1) *N is a normal subset of L.*

(2) *$N \cup \{0\}$ is a convex subset of L.*

(3) *The subgroup $[N \cup \{0\}]$ of L is an l-ideal of L.*

Proof. (1) If $a \in N$, then $a \cap b = 0$ for some $b > 0$. Thus for each $c \in L$,

$0=c+(a\cap b)-c=c+a-c\cap c+b-c$, and hence $c+a-c\in N$. (2) Suppose that $0<x<n$, where $n\in N$. Then there exists $0<b\in L$ such that $0\leqq x\cap b\leqq n\cap b=0$. Therefore $x\in N$, and hence $N\cup\{0\}$ is convex. (3) Let S be the subsemigroup of L that is generated by $N\cup\{0\}$, and suppose that $0<c\leqq a$, where $a\in S$. Then $a=a_1+\cdots+a_n$, where the a_i belong to N, and [2, p. 245] $c=c_1+\cdots+c_n$, where $0\leqq c_i\leqq a_i$ for $i=1,\cdots,n$. Thus the $c_i\in N\cup\{0\}$ and hence $c\in S$. Therefore S is a convex subsemigroup of L that contains 0. By Theoren 2.1, $[N\cup\{0\}]=[S]=\{a-b: a, b\in S\}$ and $[S]^+=S$. But since N is normal it follows that S is normal and hence $[S]$ is normal. Therefore $[N\cup\{0\}]$ is an l-ideal of L.

THEOREM 9.1. *Let N be the set of all nonunits of L. Then $[N]$ is an l-ideal of L, N is the set of all nonunits of $[N]$, and $L=\langle[N]\rangle$. $[N]$ is not a proper lexico-extension of an l-ideal.*

Proof. Clearly $[N]=[N\cup\{0\}]$. $[N]$ is an l-ideal by Lemma 9.2. Let N' be the set of all nonunits of $[N]$. Clearly $N'\subseteq N$. If $a\in N$, then $a>0$ and $a\cap b=0$ for some $0<b\in L$. Thus $b\in N$, and hence $a\in N'$. Therefore $N'=N$. $L=\langle[N]\rangle$ by Lemma 9.1. Finally suppose that $[N]=\langle I\rangle$ for some l-ideal I of $[N]$. Then by Lemma 9.1, $I\supseteq N'=N$, and hence $I=[N]$.

10. **L-groups that have no basis.** We first give some other examples of l-groups without bases besides the two quotient groups given in §8.

I. Let L be the additive group of all continuous real valued functions defined on the closed unit interval $[0, 1]$. Let $L^+=\{f\in L: f(x)\geqq 0$ for all $x\in[0, 1]\}$. Then L is an Archimedean l-group. There is a 1-1 correspondence between the closed subsets X of $[0, 1]$ and the convex subgroups G of L such that G corresponds to X if and only if

$$X = \{x \in [0, 1]: g(x) = 0 \text{ for all } g \in G\},$$

and

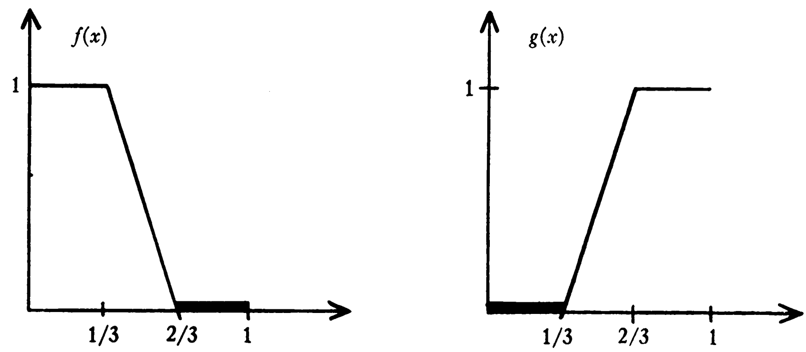
$$G = \{g \in L: g(x) = 0 \text{ for all } x \in X\}.$$

It follows that L is *cardinally indecomposable*. For if $L=G \boldsymbol{+} F$, then $[0, 1]=X\cup Y$, where X and Y are disjoint closed sets.

L does not contain any nonzero ordered convex subgroups. For if C is such a subgroup, then since L is Archimedean, C is not bounded from above. Thus by Lemma 7.1, $L=C \boldsymbol{+} D$, which is impossible. In particular, the null set is the only independent subset of L.

L is generated by its set N of nonunits. For let h be the function which is identically equal to 1 on $[0, 1]$. Then $h=f+g$, where f and g are given by the diagrams on the following page. Clearly f and g are nonunits. Thus $h\in[N]$ and by Lemma 9.2, $[N]$ is convex. Therefore $[N]=L$.

II. Let L be the set of order preserving permutations of $[0, 1]$ with resultant multiplication. Let $L^+=\{f\in L: f(x)\geqq x$ for all x in $[0, 1]\}$.



III. A cardinal sum of l-groups, one of which has no basis, has no basis.

IV. A lexico-extension of an l-group with no basis has no basis.

Note that III and IV are immediate consequence of Corollary I of Theorem 5.1. For the remainder of this section we shall *assume that L does not have a basis*. Let $\{A_\gamma: \gamma \in \Gamma\}$ be the set of all maximal convex ordered subgroups of L. In each A_γ pick a strictly positive element a_γ. Then $T = \{a_\gamma: \gamma \in \Gamma\}$ is a maximal independent subset of L.

By Theorem 5.1 there exists a strictly positive element x in L such that L^y is not an o-group for all $0 = y \leqq x$. That is, x does not exceed any basis element. In particular, $x \cap a_\gamma = 0$ for all γ in Γ. For if $x \cap a_\gamma > 0$, then $0 < x \cap a_\gamma \leqq x$ and $L^{x \cap a_\gamma}$ is an o-group. Note also that if $b \cap a_\gamma = 0$, then $b \cap c = 0$ for all $c \in A_\gamma^+$. For $b \cap c$ is an element in the o-group A_γ and if $b \cap c > 0$, then $0 < a_\gamma \cap b \cap c = 0 \cap c = 0$. Let

$$B = \{x \in L: x \cap a_\gamma = 0 \text{ for all } \gamma \text{ in } \Gamma\}$$
$$= \Big\{x \in L: x \cap \Big(\sum_{\gamma \in \Gamma} + A_\gamma\Big)^+ = 0\Big\}.$$

LEMMA 10.1. *$0 \neq B = \{x \in L^+: x$ does not exceed any basic element of $L\}$. $[B]$ is an l-ideal of L that contains no convex o-subgroups except $\{0\}$, and every other l-ideal of L with this property is contained in $[B]$.*

Proof. Let $C = \{x \in L^+: x$ does not exceed any basic element of $L\}$. We have shown that $B \supseteq C \neq \{0\}$. Suppose (by way of contradiction) that $x \in B \backslash C$. Then there exists an element y in L such that $0 < y \leqq x$ and L^y is ordered. By Lemma 3.1, $L^y \cap A_\gamma = 0$ or $L^y \subseteq A_\gamma$ or $A_\gamma \subseteq L^y$ for each γ in Γ. If $A_\gamma \subseteq L^y$ or $L^y \subseteq A_\gamma$, then $0 < a_\gamma \cap y \leqq a_\gamma \cap x = 0$. Thus $L^y \cap A_\gamma = 0$ for all $\gamma \in \Gamma$, and hence $T \cup \{y\}$ is an independent subset of L, but this contradicts the maximality of T. Therefore $B = C$.

By Theorem 5.3, $\sum_{\gamma \in \Gamma} + A_\gamma$ is an l-ideal of L. Thus by Lemma 6.2, $[B]$ is an l-ideal of L. Since B contains no basic elements, $[B]$ contains no convex o-subgroups except $\{0\}$. Finally let D be a nonzero convex subgroup of L

that contains no convex o-subgroups except $\{0\}$. Consider $0<d\in D$. If $d\cap a_\gamma>0$, then $d\cap a_\gamma\in D\cap A_\gamma$ and $D\cap A_\gamma$ is a nonzero convex o-subgroup of D. Therefore $d\cap a_\gamma=0$ for all γ in Γ, and hence $d\in B$. Thus $D\subseteq[B]$.

Let $B^*=\{x\in L^+: x\cap B^+=0\}$. Then clearly $[B^*]\supseteq\sum\boldsymbol{+}A_\gamma$.

THEOREM 10.1. *$[B^+\cup B^*]=[B]\boldsymbol{+}[B^*]$ and $[B]\boldsymbol{+}[B^*]$ is an l-ideal of L that is independent of the particular choice of T. T is a basis for $[B^*]$.*

Proof. The first part follows from the fact that $[B]$ is an l-ideal and Lemma 6.2. $[B^*]$ has a basis, for otherwise by Theorem 5.1 there exists a strictly positive element x in B^* that does not exceed a basic element, and hence by Lemma 10.1, $x\in B\cap B^*=0$. T is a maximum independent subset of $[B^*]$ and hence by Theorem 5.1, T is a basis of $[B^*]$.

Note that if $[B^*]=0$, then $L=[B]$. Thus L has no basic elements. Suppose that $[B^*]\neq 0$. Let N be the set of nonunits of L. Then clearly $[B]\boldsymbol{+}[B^*]\subseteq[N]\subseteq L$ and by Theorem 9.1, $L=\langle[N]\rangle$. The problem then is to find conditions on L so that $[N]=[B]\boldsymbol{+}[B^*]$. The following example shows that this need not always hold. Let U be a small cardinal sum of o-groups and let $V=W$ be the group of all continuous real valued functions on $[0, 1]$ (Example I in §8). Let $L=\langle U\boldsymbol{+}V\rangle\boldsymbol{+}W$, where the indicated lexico-extension is nontrivial. Then $[B^*]=U$, $B=V\boldsymbol{+}W$ and $[N]=L$.

As in Theorem 6.2 let $F=\{x\in L^+: x$ exceeds at most a finite number of disjoint elements$\}$, and consider $f\in F$. If there exists $0<b\in B$ such that $f\cap b>0$, then $f\geqq f\cap b$ and hence $f\cap b\in F\cap B$. But by Corollary II of Theorem 5.2, $f\cap b$ exceeds a basic element and by Lemma 10.1, it does not. Thus $f\cap b=0$ and hence $F\subseteq B^*$. Since $\sum\boldsymbol{+}A_\gamma$ is the basis group for $[B^*]$, it follows as in Theorem 6.2 that $[F]$ is an l-ideal of $[B^*]$ and a small lexico-sum of the A_γ.

11. **Commutative L-groups.** Throughout this section we shall assume that L is a commutative l-group. In particular, L is torsion free. Therefore there exists a unique (to within an isomorphism) *d-closure* D of L. That is, D is a torsion free divisible ($nD=D$ for all $n>0$) abelian group, $D\supseteq L$, and for each d in D there exists a positive integer n such that $nd\in L$. D is a rational vector space, and there is a unique way of extending the partial order of L to a partial order of D so that D is an l-group and L is a sublattice of D. Simply let $\{x\in D: nx\geqq 0$ in L for some $n>0\}$ be the set of positive elements of D. In particular, if L is an o-group, then so is D. As an example, if L satisfies the chain conditions (see the corollary to Theorem 7.2), then D is a small cardinal sum of isomorphic copies of the additive group of rational numbers.

For each subgroup B of L, let

$$\overline{B}=\{x\in D: nx\in B \text{ for some positive integer } n\}.$$

Then $\overline{B}$ is the d-closure of B in D. The mapping $B\rightarrow\overline{B}$ is a mapping of the set of all subgroups of L onto the set of all divisible subgroups of D that induces a 1-1 mapping of the set of all convex subgroups of L onto the set of all convex subgroups of D. In particular, each convex subgroup of D is divisible.

PROPOSITION 11.1. *If B and C are convex subgroups of L and if $L=B\oplus C$, then $D=\overline{B}\boxplus\overline{C}$.*

Proof. If $x\in\overline{B}\cap\overline{C}$, then $nx\in B\cap C=0$ for some $n>0$. Thus $x=0$, and $[\overline{B}\cup\overline{C}]=\overline{B}\oplus\overline{C}$. But since $\overline{B}$ and $\overline{C}$ are convex $\overline{B}\oplus\overline{C}=\overline{B}\boxplus\overline{C}$ by Corollary I of Theorem 2.1. If $x\in D$, then $nx\in L=B\oplus C$ for some $n>0$. Thus $nx=b+c=n\bar{b}+n\bar{c}=n(\bar{b}+\bar{c})$, where $b\in B$, $c\in C$, $\bar{b}\in\overline{B}$ and $\bar{c}\in\overline{C}$. Thus since D is torsion free, $x=\bar{b}+\bar{c}\in\overline{B}\boxplus\overline{C}$, and hence $D=\overline{B}\boxplus\overline{C}$.

PROPOSITION 11.2. *If $L=\langle B\rangle$ for some convex subgroup B of L, then $D=\langle\overline{B}\rangle$. Moreover, $D=\overline{B}\oplus Q$, where Q is an o-subgroup of D and an element $b+q$ with $b\in\overline{B}$ and $q\in Q$ is positive if and only if $q>0$ or $q=0$ and $b\geqq 0$. Q is o-isomorphic to the d-closure of the o-group L/B.*

Proof. Consider $0<x\in D\backslash\overline{B}$ and $y\in\overline{B}$. There exists a positive integer n such that $nx\in L\backslash B$ and $ny\in B$. Thus since $L=\langle B\rangle$, $nx>ny$ and hence $x>y$. Thus if $B\neq 0$, then $D=\langle\overline{B}\rangle$ (see [4, Lemma 1.1]). If $B=0$, then $\overline{B}=0$ and L is an o-group. Thus D is also an o-group and hence $D=\langle\overline{B}\rangle$. Since $\overline{B}$ is divisible, $D=\overline{B}\oplus Q$ for some divisible subgroup Q of D. Consider $x=b+q\in\overline{B}\oplus Q$. If $q>0$, then $0<q\in D\backslash\overline{B}$, and hence $q>\overline{B}$. Therefore $q>-b$ and hence $x=b+q>0$. If $x>0$ and $q\neq 0$, then $\overline{B}+x=\overline{B}+q$ is positive, and so by Lemma 9.1, $q>0$. Thus $b+q$ is positive if and only if $q>0$ or $q=0$ and $b\geqq 0$.

$$Q\cong D/\overline{B}\supseteq (L+\overline{B})/\overline{B}\cong L/(L\cap\overline{B}) = L/B.$$

Clearly $D/\overline{B}$ is the d-closure of $(L+\overline{B})/\overline{B}$. It follows that Q is o-isomorphic to the d-closure of L/B, and hence Q is an o-group.

Note that we have shown that if $D=\langle S\rangle$ for some convex subgroup S of D, then D is a *direct lexico-extension* of S. That is, $D=S\oplus T$ and $s+t\in S+T$ is positive if and only if $t>0$ or $t=0$ and $s\geqq 0$. Therefore the structure of an abelian divisible l-group that satisfies (F) is given completely by Theorem 6.1. For in this case all the *lexico-extensions used are direct.*

PROPOSITION 11.3. *If L satisfies* (F), *then so does D.*

Proof. Suppose (by way of contradiction) that there exists an element d in D that exceeds the disjoint subset $\{x_i: i=1, 2, \cdots\}$ of D. There exists a positive integer n such that $nd\in L$. Thus by Corollary III of Theorem 5.2, L^{nd} has a finite basis, say $a_1, \cdots, a_k$. But there exists a positive integer m such that $m(nd)>(mn)x_i$ and $(mn)x_i\in L$ for $i=1, \cdots, k+1$. Since the $(mn)x_i$ are disjoint and belong to L^{nd}, we have a contradiction.

Thus if L satisfies (F), *then the structure of D is given completely by Theorem* 6.1.

PROPOSITION 11.4. *Let $U=\{d_\gamma:\gamma\in\Gamma\}$ be a basis for D, and for each d_γ pick a positive integer n_γ such that $n_\gamma d_\gamma\in L$. Then $V=\{n_\gamma d_\gamma:\gamma\in\Gamma\}$ is a basis for L. Conversely each basis for L is also a basis for D. In particular L has a basis if and only if D has a basis.*

Proof. Clearly $L^{n_\gamma d_\gamma}\subseteq D^{d_\gamma}$ for each γ in Γ. Each D^{d_γ} is an o-group, and hence each $n_\gamma d_\gamma$ is basic in L. Also it follows easily that V is a maximal disjoint subset of L, and hence V is a basis for L.

Let $S=\{a_\gamma:\gamma\in\Gamma\}$ be a basis for L. Then each L^{a_γ} is a convex o-subgroup of L and hence $D^{a_\gamma}=\overline{L^{a_\gamma}}$ is a convex o-subgroup of D. Thus each a_γ is basic in D, and by an easy argument S is a maximal disjoint subset of D. Therefore S is a basis for D.

Note that if L is Archimedean and has a basis, then D is also Archimedean and has a basis. Hence by Theorem 7.2, D can be embedded in a large cardinal sum of divisible subgroups of the additive group of real numbers.

We have applied our structure theorems to the rational vector space D. These theorems can also be applied to vector lattices over ordered division rings provided that we restrict our attention to convex subspaces.

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